Second-order accuracy of two-time-level semi-Lagrangian schemes

By I. G. GOSPODINOVA, V. G. SPIRIDONOV\textsuperscript{1} and J.-F. GELEYN\textsuperscript{2}

\textsuperscript{1}National Institute of Meteorology and Hydrology, Bulgaria
\textsuperscript{2}Météo-France/CNRM/GMAP, France

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SUMMARY

The accuracy of the two-time-level semi-Lagrangian semi-implicit time integration scheme is analysed. Two relatively independent problems are identified—the discretization of the implicit trajectory equation and the treatment of the nonlinear residual. Two theorems giving conditions for second-order accuracy of the two-time-level semi-Lagrangian semi-implicit scheme are stated. As a result, a new scheme for the trajectory equation is proposed and a class of schemes for the nonlinear residual, which are second-order accurate in time, are introduced. A simplified one-dimensional ‘shallow water’ model is developed in order to test the proposed scheme in comparison with other, previously known, ones. Some experiments proving stability, accuracy and conservation ability are presented.

KEYWORDS: Accuracy Conservation Numerical weather prediction Semi-Lagrangian Stability

1. Introduction

The semi-Lagrangian (SL) semi-implicit (SI) method proposed by Robert (1981, 1982) has been investigated by many authors. The early versions of the method were three-time-level schemes since they were built on the basis of the previously existing Eulerian ‘leap frog’ schemes. Staniforth and Côté (1991) provide a comprehensive review of the early SL schemes as well as many developments and studies related to their stability, accuracy and conservation properties. Later on, the first two-time-level (2TL) schemes defined in McDonald (1986) and Temperton and Staniforth (1987) have proved to be more efficient and have been implemented into several operational models (e.g. McDonald and Haugen 1992, 1993; Yessad 1996). One of the most important features of the early 2TLSL schemes was the extrapolation in time of the wind velocity to obtain the position of the middle point of the SL trajectory. Temperton and Staniforth (1987) applied a linear extrapolation in time. We will call this technique Classical 2TLSL. Côté and Staniforth (1988) examined a quadratic extrapolation involving three time levels. Some discussions on the Classical 2TLSL can also be found in Staniforth and Côté (1991). Already, at that time, the extrapolation technique was recognized as a potential source of instability. However, no problems were reported at the time and the scheme was adopted worldwide. In some recent cases, characterized generally by strong wind gradients, spurious oscillations have been obtained with a number of 2TLSL models. A typical example is the famous ‘Baltic Jet’ case (see Fig. 1), where some non-meteorological noise was observed in the core of a strong mid-troposphere jet and was reported by almost all European 2TLSL models integrated over the east Atlantic region (e.g. McDonald 1998a). In Fig. 1 the instability can be seen in the relative-humidity field as a wave extending over the Baltic Sea region. The example is from the ALADIN/LACE† limited-area model. The extrapolation in time for obtaining the middle point of the SL trajectory was identified as a source of the instability (e.g. Hortal 1998b; McDonald 1998a). Geleyn, Bubnová, and Spiridonov (1997, personal communication) examined a number of extrapolation techniques in

* Corresponding author: Météo-France/CNRM/GMAP, 42, Av. G. Coriolis, 31057 Toulouse-CEDEX 1, France.
\dagger ALADIN is the acronym for the limited-area high-resolution model of Météo-France and partners (see Members of the ALADIN international team 1997). LACE—Limited Area Central European model. For more details contact R. Bubnová—project leader, CHMI, 17, Na Sabatce, Prague 4, 14306 Czech Republic.
order to resolve the instability problem. Hortal (1998a,b) studied the problem, and
developed and examined a new scheme which avoided the extrapolation and only
involved quantities from exact time steps in the displacement computation. Hortal's
scheme seemed to be successful in solving the noise problems and was adopted in all
models linked to the European Centre for Medium-Range WeatherForecasts Integrated
Forecasting System (ECMWF/IFS). In his technical note Hortal (1998a) discussed the
possibility of using the explicit forcing term which appeared on the right-hand side
(r.h.s.) of the horizontal-momentum equation as an acceleration for the solution of the
trajectory equation. However, he abandoned this approach in order to remain consistent
with the vertical part of his scheme. The problem is that in a hydrostatic atmospheric
model there is no corresponding prognostic equation for the vertical velocity. The
importance of including the explicit forcing term from the r.h.s. of the momentum
equation in the trajectory calculation has already been pointed out by Spiridonov (1991).

Other problems of the SL technique are related to its interaction with orography.
Similar oscillations to those in the 'Baltic Jet' case can often be observed in the vicinity
of mountains. Coifler et al. (1987) studied this problem. The treatment of the nonlinear
residual of the forcing term in the momentum equation was considered to be a main
source of such oscillations. Tanguay et al. (1992), Rivest and Staniforth (1994) and
Ritchie and Tanguay (1996) examined the technique of spatial averaging which was
proven to reduce such noise and was adopted. As a result, the version of the Classical
scheme for the nonlinear residual, examined in this work, was invented. It consists of a linear extrapolation in time of the nonlinear term and its averaging along the SL trajectory. Recently, Hortal (1998a,b) contributed to the solution of the problem by implementing his Stable Extrapolation Two Time Level Scheme (SETTLS), also for the treatment of the nonlinear residual. McDonald (1998b) also studied the instability caused by the nonlinear terms and examined the effect of the de-centring and the spatial-averaging techniques.

In this work we make an attempt to study the accuracy of the 2TLSL SI scheme analytically. In section 2 we show that a wide class of 2TLSL schemes are equivalent if the departure point is properly calculated, using the explicit acceleration term. In this case, the Classical and SETTLS schemes for the nonlinear residual are equivalent and belong to the same class. A simplified one-dimensional (1D) model is presented in section 3. It was built on the basis of the 'shallow water' equations and contains all the basic features which characterize the dynamics of modern SL models, such as spectral representation of real fields, control of stability with a semi-implicit scheme, etc. The 1D model was used in a special form, so that the nonlinear term appears on the r.h.s. of the momentum equation while the r.h.s. of the continuity equation is linear. This version of the simplified model corresponds to the real numerical weather prediction (NWP) models, where the orography contributes mainly through the momentum equation. It allowed us to test the interaction with orography of a variety of 2TLSL schemes. The second-order accuracy of the 2TLSL discussed in section 2 is illustrated with experiments presented in section 4. Some discussions on possible benefits of the 2TLSL NWP models are given at the end of the paper.

2. THEORETICAL ASPECTS OF THE SECOND-ORDER ACCURACY OF TWO-TIME-LEVEL, SEMI-IMPLICIT, SEMI-LAGRANGIAN SCHEMES

(a) A class of two-time-level, semi-Lagrangian schemes which are second-order accurate in time in the case of exact trajectories

Let us consider the 1D forced-advection equation:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = R(t, x) = L + N$$

with a solution $u_0$ and r.h.s. expressed as $R = N + L$, where $L$ is the major linear part of $R$, and $N$ is a small nonlinear residual. We have adopted the traditional SL notation in which the superscripts '−', '0' and '++' stand for the $t - 2\Delta t$, $t - \Delta t$ and $t$ time levels respectively ($t$ being the forecasted one) and the subscripts 'D' and 'A' stand for the departure ($x - \Delta x$) and arrival ($x$) points of an SL trajectory. For example, the nonlinear residual at time $t - 2\Delta t$ and at position $x - \Delta x$ will appear as $N_D^{−}$. We will examine a group of two-time-level discretizations generally described by:

$$\frac{u_A^+ - u_A^0}{\Delta t} = a_1 N_A^{0} + a_2 N_A^{-} + a_3 N_D^{0} + a_4 N_D^{-} + \frac{1}{2}(L_A^{+} + L_D^{0})$$

where $a_1, a_2, a_3, a_4$ and $a_4$ are arbitrary constants. As one can see, this is a 2TL scheme with an implicitly treated linear part. We assume exact SL trajectories or, in other words, we assume that we know the departure points exactly. What remains to be defined is the ensemble of arbitrary constants, $a_i$. Typically in a 2TLSL model we have available the result from the last two time steps and we evaluate all quantities at arrival or departure points. Thus, with Eq. (2) we cover all possible combinations. We will find some
relationships between the constants, $a_i$, resulting from the condition of second-order accuracy of Eq. (2). The left-hand side (l.h.s.) of Eq. (2), expanded as a Taylor series, reads:

$$\text{l.h.s.} = \frac{1}{\Delta t} \{ u(t, x) - u(t - \Delta t, x - \Delta x) \} = -\frac{1}{\Delta t} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \frac{d^k u}{k!}, \quad (3)$$

where

$$d^k u = \Delta t^k \left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \right)^k u \quad \text{and} \quad u = u(t, x). \quad (4)$$

Thus, by keeping only the terms up to order $O(\Delta t)$ for the l.h.s. we obtain:

$$\text{l.h.s.} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - \frac{\Delta t}{2} \left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \right) \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) + O(\Delta t^2). \quad (5)$$

Let the solution of Eq. (2) ($u_A^+$ in Eq. (2)) or $u = u(t, x)$ in Eqs. (3), (4), and (5) be considered as an approximation of the solution of Eq. (1), $u_0$, represented by the sequence:

$$u = u_0 + \Delta t u_1 + \Delta t^2 u_2 + \cdots, \quad (6)$$

where $\Delta t^i u_i$ are remainders. By replacing $u$ in Eq. (5) by Eq. (6) and keeping the same order of accuracy we find:

$$\text{l.h.s.} = \frac{\partial u_0}{\partial t} + u_0 \frac{\partial u_0}{\partial x} + \Delta t \left( \frac{\partial u_1}{\partial t} + \frac{\partial u_1 u_0}{\partial x} \right)\right) - \frac{\Delta t}{2} \left( \frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x} \right) \left( \frac{\partial u_0}{\partial t} + u_0 \frac{\partial u_0}{\partial x} \right) + O(\Delta t^2). \quad (7)$$

After taking into account that $u_0$ is the solution of Eq. (1) and using that

$$\frac{\partial R}{\partial t} + u \frac{\partial R}{\partial x} = \frac{dR}{dt}, \quad (8)$$

we can rewrite Eq. (7) in the form:

$$\text{l.h.s.} = R - \frac{\Delta t}{2} \frac{dR}{dt} + \Delta t \left( \frac{\partial u_1}{\partial t} + \frac{\partial u_1 u_0}{\partial x} \right) + O(\Delta t^2). \quad (9)$$

For the r.h.s. of Eq. (2), by applying the same technique as in Eqs. (3) and (4) and keeping the same order of accuracy, one could write:

$$\text{r.h.s.} = (a_1 + a_2 + a_3 + a_4) N + L$$

$$- (a_1 + 2a_2 + a_3 + 2a_4) \Delta t \frac{\partial N}{\partial t} - \frac{\Delta t}{2} \frac{\partial L}{\partial t}$$

$$- (a_3 + a_4) \Delta t u \frac{\partial N}{\partial x} - \frac{\Delta t}{2} u \frac{\partial L}{\partial x} + O(\Delta t^2). \quad (10)$$

By comparing Eq. (9) and Eq. (10), trying to find similar terms, and keeping in mind that $R = N + L$, we can obtain the following relations:

$$a_1 + a_2 + a_3 + a_4 = 1,$$

$$a_1 + 2a_2 + a_3 + 2a_4 = \frac{1}{2},$$

$$a_3 + a_4 = \frac{1}{2}. \quad (11)$$
The equation system (11) defines $a_1, a_2, a_3$ and $a_4$ up to an arbitrary parameter. By choosing

$$a_1 + 2a_2 = \alpha + \frac{1}{4}$$

(12)

we obtain:

$$a_1 = \frac{3}{4} - \alpha, \quad a_2 = \alpha - \frac{1}{4},$$

$$a_3 = \frac{3}{4} + \alpha, \quad a_4 = -\alpha - \frac{1}{4}.$$  

(13)

Now, by replacing $a_i$ in Eq. (10) with Eq. (13), we obtain:

$$\text{r.h.s.} = R - \frac{\Delta t}{2} \left( \frac{\partial R}{\partial t} + u \frac{\partial R}{\partial x} \right) + O(\Delta t^2) = R - \frac{\Delta t}{2} \frac{dR}{dt} + O(\Delta t^2).$$

(14)

Comparing again Eq. (9) with Eq. (14) we find that if

$$\frac{\partial u_1}{\partial t} + \frac{\partial u_1 u_0}{\partial x} = 0,$$

(15)

the order of approximation of Eq. (2) will not be less than two. The Cauchy problem with respect to $u_1$ for the homogeneous Eq. (15) with zero boundary and initial conditions has a unique trivial solution, $u_1 = 0$ (e.g. Smirnov 1981).

In this way we prove the following:

**Theorem 1:** All discretizations of Eq. (1) of the kind:

$$\frac{u_A^+ - u_D^0}{\Delta t} = \frac{3}{4} (N_A^0 + N_D^0) - \frac{1}{4} (N_A^- + N_D^-)$$

$$+ \alpha (-N_A^0 + N_A^- + N_D^0 - N_D^-)$$

$$+ \frac{1}{2} (L_A^+ + L_D^0)$$

(16)

are equivalent to second-order accuracy for any $\alpha$.

Setting $\alpha = 1/4$ we arrive at a SETTLS type of approximation. With $\alpha = 0$ we have the Classical type (see section 4).

(b) Accuracy depending on departure point

For the proof of our first theorem, we have assumed that the exact positions of the departure points are known. In this subsection, we will define the residual order of accuracy resulting from the approximation of the displacement equation. As it was shown in Gospodinov and Spiridonov (1999, 2000), an exact solution of the forced-advection equation (1) can be written but the acceleration or the forcing term needs to be taken into account in the trajectory computations. Following Gospodinov and Spiridonov (1999), the exact solution of the equation of motion reads:

$$\delta = tu(t, x) - \int_0^t \tau R \, d\tau,$$

(17)

where $\delta$ (or $\Delta x$) is the displacement, $R$ is the forcing term and the initial time has been chosen to be 0. By rewriting it in implicit form and approximating the integral we obtain:

$$\delta = \Delta t \left( u_D^0 + \frac{\Delta t}{2} R \right),$$

(18)
where \( \tilde{R} \) is an approximation of \( R \) and \( t \) is replaced by \( \Delta t \). Using again the Taylor series expansion of \( u^0_D \) and keeping second-order accuracy we arrive at:

\[
\begin{align*}
u^0_D &= u - \Delta t \frac{\partial u}{\partial t} - \Delta t \left( \frac{u^0_D}{2} + \frac{\Delta t}{2} \left( \tilde{R} - R \right) \right) \frac{\partial u}{\partial x} \\
&\quad + \frac{\Delta t^2}{2} \frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^2}{2} \left( u^0_D \right)^2 \frac{\partial^2 u}{\partial x^2} + \Delta t^2 u^0_D \frac{\partial^2 u}{\partial t \partial x} + O(\Delta t^3). \tag{19}
\end{align*}
\]

From here, by iteratively replacing \( u^0_D \) on the r.h.s. and keeping the order of accuracy we get:

\[
\frac{u - u^0_D}{\Delta t} = R - \frac{\Delta t}{2} \frac{dR}{dt} + \frac{\Delta t}{2} \left( \tilde{R} - R \right) \frac{\partial u}{\partial x} + O(\Delta t^2), \tag{20}
\]

where Eq. (8) has been used. Taking into account Eq. (14) one can see that the first two terms from the r.h.s. are identical and the residual up to second-order accuracy is:

\[
\frac{\Delta t}{2} \left( \tilde{R} - R \right) \frac{\partial u}{\partial x}. \tag{21}
\]

Now we can formulate:

**Theorem 2:** The second-order accuracy of the class of schemes defined by Eq. (16) in Theorem 1 is determined by the residual, Eq. (21).

Thus, if we choose an approximation of \( R \) which is first-order accurate in time, we arrive at second-order accuracy for Eq. (2). The first-order accuracy in time of the approximation of \( R \) means that we can assume it is constant along the trajectory and, thus, it is enough to evaluate it at the departure point only.

The accuracy of the semi-Lagrangian method has also recently been considered by Durran (1999).

### 3. Description of the Simplified Model

The model we have used for testing was developed on the basis of the 'shallow water' equations reduced to one dimension.

**(a) Classical one-dimensional ‘shallow water’ model**

The two equations of the classical 'shallow water' model in Lagrangian form read:

\[
\begin{align*}
\frac{du}{dt} &= -\Phi_x = A_u, & \frac{d(\Phi - \Phi_s)}{dt} &= -((\Phi - \Phi_s)u_x) = A_\Phi, \tag{22}
\end{align*}
\]

where \( \Phi \) is the free-surface geopotential, \( \Phi_s \) is the bottom or the orography geopotential, and \( u \) is the wind velocity (see Fig. 2). \( A_u \) and \( A_\Phi \) are a generalized notation for the r.h.s. of both equations in Eq. (22). In this form the continuity equation is nonlinear and contains the orography term. Some simple developments show that equation system (22) has the following invariants:

\[
\begin{align*}
\frac{\partial}{\partial t} \frac{1}{H} \int_H (\Phi - \Phi_s) \, dx &= 0, & \frac{\partial}{\partial t} \frac{1}{H} \int_H u \, dx &= 0, \\
\frac{\partial}{\partial t} \frac{1}{H} \int_H \{(\Phi - \Phi_s)u^2 + (\Phi^2 - \Phi_s^2)\} \, dx &= 0, \tag{23}
\end{align*}
\]
if periodic boundary conditions are assumed (Gospodinov and Spiridonov 1999, 2000). In Eq. (23) $H$ stands for any characteristic domain dimension. The three invariants in Eq. (23) correspond respectively to mass, momentum and total mechanical-energy conservation. The equation system (22) also has a non-trivial stationary solution which obeys the following laws:

$$ u(\Phi - \Phi_s) = \text{const}, \quad \frac{u^2}{2} + (\Phi - \Phi_s) = \text{const}. \quad (24) $$

The most important feature of our model is the semi-implicit scheme. Generally the r.h.s. of the equations is divided into a linear part, $L$, and a nonlinear residual, $N$. The linear term, $L$, is treated implicitly (see section 2). In discretized form the equation system (22) reads:

$$ u_A^+ - u_D^0 = (L_{u_A^+} + L_{u_D^0}), \quad \Phi_A^+ - \Phi_D^0 = f(N_{\Phi}) + (L_{\Phi_A^+} + L_{\Phi_D^0}), \quad (25) $$

where

$$ L_u = \frac{\Delta t}{2} A_u, \quad L_\Phi = -\frac{\Delta t}{2} \Phi_{u_x}, \quad \Phi = \frac{1}{K} \sum_{k=1}^{K} \Phi_k \quad \text{and} \quad N_\Phi = \Delta t A_\Phi - 2L_\Phi. $$

The traditional notation for the semi-Lagrangian discretization was used (see section 2). The function $f(X)$ in Eq. (25) generalizes the treatment of the nonlinear residual. The semi-implicit scheme leads to a Helmholtz equation for the wind velocity which is easily solved in spectral space.

Another important feature of a semi-Lagrangian model is the implicit-in-space computation to obtain the position of the departure point, $D$. It consists of solving the equation of motion, $dx/dt = u$, which generally leads to the following discretization in time:

$$ \delta = x_A - x_D = \Delta t U. \quad (26) $$

The symbol $U$ generalizes any particular scheme for the trajectory computations and has to be examined and studied in detail (see section 4).

(b) Full ‘shallow water’ model with a nonlinear momentum equation

In a number of current three-dimensional (3D) SL SI models, the orography contributes mainly through the momentum equation in the form of two terms cancelling each other in the case of a motionless stratified state of the atmosphere (see Ritchie et al. 1995; Simmons and Burridge 1981). An appropriate change of variables leads to the following form of equation system (22):

$$ \frac{d\Psi}{dt} = -e^\Psi \Psi_x - \Phi_{s_x} = A_u, \quad \frac{d\Psi}{dt} = -u_x = A_\Psi, \quad (27) $$

where $\Psi = \ln(\Phi - \Phi_s)$ (see Fig. 2). In this form the continuity equation is linear and the momentum equation contains the orography term in a very similar way to that in the 3D model. The invariants defined within Eq. (23) and the stationary solution Eq. (24) remain valid where $\Phi$ must be replaced by $(e^\Psi + \Phi_s)$. Only small technical changes for the semi-implicit computations are required. The trajectory computations are identical to those of the Classical model.
Figure 2. Geometry and parameters of the one-dimensional 'shallow water' model. $\Phi$—geopotential; $\Phi$—mean geopotential; $u$—wind velocity; $\Phi_0$—surface geopotential; $H$—domain dimension; $\Psi$ contains the geopotential after the change of variables.

(c) Linear model

The linear version of Eq. (22) reads:

$$\frac{du}{dt} = -\Phi_x, \quad \frac{d\Phi}{dt} = -\Phi u_x. \quad (28)$$

The linear version of Eq. (27) reads:

$$\frac{du}{dt} = -\Phi \Psi_x, \quad \frac{d\Psi}{dt} = -u_x, \quad (29)$$

where the orography terms have been omitted (a flat-bottom assumption). The linear version of the model does not have any problem with the treatment of the nonlinear residual since the r.h.s. of both equations is linear and implicitly treated. Thus, it provides us with a pure environment for studying the second-order accuracy of SL trajectory discretizations. The mass and momentum conservation laws from Eq. (23) are valid for the linear model as well.

(d) Domain parameters, initialization and other features

A regular grid of 200 equidistant points with $\Delta x = 5 \times 10^4$ m was used. A time step of $2 \times 10^3$ s was used which, combined with the mean wind velocity, exceeds the Courant–Friedrich–Lévy limit by about 2.5 times. For all experiments, 1000 time steps were executed in order to allow the examined schemes to reveal their conservation ability. Cubic interpolation was used, although originally, in the actual 3D NWP models, some linear interpolation took place. We have applied ten instead of two iterations for
Figure 3. Initial fields, used for the experiments from Group I and Group II. Left vertical axis—geopotential ($\Phi$) in m$^2$s$^{-2}$; right vertical axis—wind velocity ($u$) in m s$^{-1}$. Horizontal axis—model domain in grid-point number. Full line—geopotential; dashed line—mean geopotential; dotted line—homogeneous wind field.

Figure 4. Initial fields and geometry for the experiments from Group III. Axes, full and dotted lines as in Fig. 3; short-dashed line—$u(\Phi - \Phi_0)$ product initial field; long-dashed line—orography (scaled).
solving the implicit displacement equation. The last two features practically eliminate any problems that might be related to the interpolation or the iteration techniques.

Three different groups of experiments are presented:

Group I—mainly experiments with the model as it has been described in subsection 3(c) with a flat bottom. The initial geopotential field is periodic and was randomly chosen (see Fig. 3). Its mean value is \(10^5\ \text{m}^2\text{s}^{-2}\) in most cases. The initial wind field is homogeneous with a mean value of 60 m s\(^{-1}\) (see Fig. 3).

Group II—tests with the full model (subsection 3(a)) without orography and initialized as in the experiments of Group I.

Group III—tests with the full model but with orography (subsection 3(b)) and initialized with an experimentally obtained stationary solution. Figure 4 presents the initial geopotential and wind-velocity fields, the initial field of the product \(u(\Phi - \Phi_s)\) from the first law in Eq. (24), and the position of the mountain. The maximum height of our mountain is about 5\% of the mean free-surface geopotential although it looks higher in the figure.

4. Numerical Tests with the Simplified Model

(a) The trajectory equation schemes

In order to study the SL trajectory schemes, we have chosen the linear model described in section 3(c). The results given in this subsection are obtained with the Classical version where the true free-surface geopotential is a prognostic variable. However, the majority of the experiments were performed again with the other version (with \(\Psi\) as a prognostic variable) and the results were found to be practically identical.

As was shown, the linear 'shallow water' system of equations obeys the mass and momentum conservation laws. Thus, we seek a scheme which is mass and momentum conserving to the order of accuracy of the applied interpolation and iteration techniques. Four different schemes were examined:

Classical SL trajectory scheme—applies linear extrapolation in time of the wind-velocity field to moment \(t - (\Delta t/2)\) using the last available two time steps. An interpolation at the middle point of the SL trajectory takes place and the value obtained for the wind velocity is used to recalculate the next guess for the displacement (see Staniforth and Côté 1991). The implicit displacement equation (26) for the Classical scheme reads:

\[
\delta^{(v)} = \Delta t \left\{ \frac{1}{2} u(t - \Delta t, x - \frac{1}{2} \delta^{(v-1)}) - \frac{1}{2} u(t - 2\Delta t, x - \frac{1}{2} \delta^{(v-1)}) \right\}. \tag{30}
\]

There is also another version of the Classical scheme which was developed at ECMWF before the SETTLS one. It applies spatial averaging of the time-extrapolated wind and corresponds here to the Classical scheme for the treatment of the nonlinear residual, Eq. (34).

SETTLS SL trajectory scheme—assumes uniformly accelerating motion along the SL trajectory. The acceleration term is constructed by using the historical wind fields from the last available two time steps (see Hortal 1998a,b). Equation (26) for the SETTLS scheme reads:

\[
\delta^{(v)} = \Delta t \left\{ u(t - \Delta t, x - \delta^{(v-1)}) + \frac{1}{2} u(t - 2\Delta t, x) - u(t - 2\Delta t, x - \delta^{(v-1)}) \right\}. \tag{31}
\]

Following the theory presented in section 2, we have developed and tested the following two schemes:
Test 1—uniform motion is assumed along the SL trajectory. Therefore, it is enough to assess the particle’s velocity on departure point only. Equation (26) in this case reads:

$$\delta^{(v)} = \Delta t u(t - \Delta t, x - \delta^{(v-1)}).$$  \hfill (32)

The scheme provides a first-order accuracy.

Test 2—uniformly accelerating motion is assumed along the SL trajectory. Therefore, it is enough to assess the particle’s acceleration at the departure point only. Equation (26) in this case reads:

$$\delta^{(v)} = \Delta t \left(u + \frac{\Delta t}{2} A_u\right) (t - \Delta t, x - \delta^{(v-1)}),$$  \hfill (33)

where $A_u$ denotes the acceleration which naturally results from Eq. (22). The scheme provides second-order accuracy (see section 2(b)).

Figure 5 presents the trends of the mean geopotential and the mean velocity, obtained after integration of the linear classical model with the four trajectory schemes introduced above (experiments from Group 1). The upper four lines are the geopotential trends and the lower three lines are the wind-velocity trends (the two obtained with Test 1 and Test 2 schemes coincide). One can see that both Classical and SETTLS schemes are neither mass nor momentum conserving, although the SETTLS one is better for momentum conservation. The Test 1 (uniform motion) scheme is already momentum conserving but still there is some loss of mass. Only the Test 2 (uniformly accelerating
motion) scheme is both mass and momentum conserving. It is also energy conserving (not shown).

Figure 6 presents the final geopotential and wind-velocity fields, obtained through the same experiments from Group I. The upper four curves are the geopotential fields corresponding to the four trajectory tests and the lower three curves are the wind-velocity fields corresponding to the Classical, SETTLS and both Test 1 and Test 2 schemes which practically coincide. The distance between the curves gives the loss of quantity at the end of integration. It is noticeable that the SETTLS and Classical schemes produce results quite different in shape from those obtained with the other two schemes. This feature demonstrates the influence of the inexact order of accuracy. In Gospodinov and Spiridonov (1999) the same trajectory schemes, except the Test 1 one, were tested for stability with an earlier version of the same simplified model. The Test 2 scheme ensured stable integration for time steps which the other two could not withstand.

Thus, by demonstrating the conservation, stability and accuracy properties of the examined trajectory schemes we have illustrated the statement of Theorem 2 (see section 2(b)).

(b) The class of schemes for the nonlinear residual

In this subsection some experimental results supporting the statement of Theorem 1 in section 2(a) will be given. As was shown in the previous subsection, the Test 2 trajectory scheme provides the best conservation which is a demonstration of its exact second-order accuracy. For the rest of our 1D model tests, we have adopted this
trajectory scheme as being the closest one to the exact SL trajectory equation. In order to illustrate Theorem 1, we have chosen three different schemes for the treatment of the nonlinear residual:

Classical scheme—one can obtain this scheme by choosing $\alpha = 0$ in Eq. (16). It has the same concept as the Classical trajectory scheme. In addition to the linear extrapolation in time, a spatial averaging along the SL trajectory takes place. The function $f(N)$, as defined in Eq. (25), for this scheme reads:

$$f(N) = \frac{3}{4} (N_D^0 + N_A^0) - \frac{1}{4} (N_D^- + N_A^-).$$

(34)

SETTLS scheme—the choice of $\alpha = 1/4$ in Eq. (16) results in obtaining the Hortal's scheme for the nonlinear residual which also corresponds to Hortal's trajectory scheme. In this case $f(N)$ reads:

$$f(N) = N_D^0 + \frac{1}{2} (N_A^0 - N_D^-).$$

(35)

Test 3 scheme—finally, by arbitrarily choosing $\alpha = 3/4$, one can obtain a scheme which formally obeys Theorem 1 but does not have any physical meaning. Its $f(N)$ reads:

$$f(N) = \frac{3}{2} N_D^0 + \frac{1}{2} N_A^- - N_D^-.$$

(36)

The nonlinear models described in section 3 were used to obtain the mean geopotential and the mean wind-velocity trends shown in Fig. 7. The three descending lines are
the mean wind-velocity trends for the three schemes given above. After 1000 time steps
the loss of momentum is negligible and the three schemes produce very close trends.
The thick horizontal line contains the three mean geopotential trends which coincide.
The conservation of mass is very robust.

Figure 8 gives the final fields obtained through the same experiments. All geopotential
fields practically coincide (the upper curve). All wind fields practically coincide too
(lower curve).

As was discussed in the introduction, the treatment of the nonlinear term was always
related to the orography problem. Here we also present a simple experiment illustrating
the ability of the three schemes to maintain a stationary flow over orography (see Fig. 4).
The final geopotential and $u(\Phi - \Phi_s)$ product fields, obtained with experiments from
Group II, are given in Fig. 9. The bound of broken lines around the straight full line
at the top are the $u(\Phi - \Phi_s)$ product fields. The full straight line is the initial one.
The scale (which is on the left axis) is expanded in order to see the details and to
demonstrate that, again, all three schemes produce very close results. The middle U-
shaped curve contains all the final geopotential fields as well as the initial one which
practically coincide. The lowest thick full line gives the position of the mountain. Again,
all examined schemes have practically identical behaviour.

With these two groups of experiments we have proven that all schemes of the class
defined in Theorem 1 are equivalent and have similar behaviour in different situations.
The problem with the treatment of the nonlinear residual is still to be studied especially
Figure 9. Final fields, obtained with the experiments from Group III (see text). Left vertical axis—\(u(\Phi - \Phi_0)\) product (scaled); right vertical axis—geopotential in m\(^2\)s\(^{-2}\); horizontal axis—as in Fig. 3. The upper bound of curves are the \(u(\Phi - \Phi_0)\) product fields and the lower bound of curves (practically one full line) are the geopotential fields. The thick full line is the mountain (scaled).

where it interlaces with the orographic resonance problem. However, the reason for presenting these results was only to give some support to the statement of Theorem 1 in order to illustrate the quality of the uniformly accelerating motion trajectory scheme (Test 2). A more profound study of the orography problem goes beyond the scope of this work and we will not go into more details here.

5. CONCLUSIONS

The properties of the proposed trajectory scheme were examined with a simplified 1D 'shallow water' model which was used in a special form with a nonlinear momentum equation in order to better mimic the environment of a modern 3D NWP model. Robust conservation and analytically derived second-order accuracy were demonstrated. Based on the results presented, we have drawn the following conclusions:

1. The numerical scheme for the trajectory equation is crucial for the quality of the entire SL method.

2. The existence of a class of schemes for the nonlinear residual provides us with flexibility for adapting the 2TSLSL scheme to various problems without losing accuracy.

The implementation of the proposed trajectory scheme in the horizontal in the ALADIN limited-area model is automatic because there is an explicit estimate of the horizontal acceleration. However, the model is more sensitive to vertical motion than to horizontal. Thus, the vertical SL trajectory is the most important part of the entire SL scheme for
the 3D model. The definition of the vertical motion in a 3D NWP model with the hybrid vertical coordinate of Simmons and Burridge (1981), as in the ALADIN model, requires more research and is under development. Several solutions might be envisaged and the preliminary results seem to indicate that starting from the continuity equation might be the most promising way.

Prospectively we would suggest the application of the Test 2 trajectory scheme in the horizontal and the vertical (possibly with some simplification) together with the SETTLS scheme for the treatment of the nonlinear residual because of its already successful implementation in the ARPEGE/IFS and ALADIN models.

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