Simple adjoint method for three-dimensional wind retrievals from single-Doppler radar

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SUMMARY
A new simple adjoint (SA) method is developed for retrieving three-dimensional wind from single-Doppler observations. The method uses the full momentum equations and the mass continuity equation as weak constraints, in addition to the strong constraints of the radial-wind and reflectivity equations. With these weak constraints, the unknown source term in the previous two-dimensional SA method is now explicitly estimated in terms of perturbation pressure and buoyancy. This improves the wind retrievals, although the retrieved perturbation pressure and buoyancy may not be accurate. Bi-spline basis functions are used to express the retrieved fields on coarse finite-element meshes. The bi-spline representation filters short-wave noise and effectively reduces the number of unknowns that need to be retrieved, so the retrievals are further improved. The method is tested with Doppler-radar data collected during the Phoenix II field experiment. The retrieved winds are very close to the dual-Doppler observed winds and the difference is about 1.0 m s⁻¹. The retrievals can be further improved by adding smoothness constraints in the cost function, especially when the bi-spline representation is not used. The robustness of the method is examined in terms of sensitivities to the weights specified for the weak constraints in the cost function.

KEYWORDS: Dual-Doppler reflectivity Phoenix

1. INTRODUCTION

Because the Next Generation Radar (NEXRAD) network provides only single-Doppler scanning over most areas in the USA, research efforts have been made in recent years to develop various methods for the retrieval of meteorological parameters from single-Doppler data (Rinehart 1979; Wolfsberg 1987; Kapitza 1991; Liou et al. 1991; Sun et al. 1991; Qiu and Xu 1992), and substantial progress has been made in demonstrating the feasibility with real Doppler data (Tuttle and Foote 1990; Laroche and Zawadzki 1994; Sun and Crook 1994; Xu et al. 1994a,b; Shapiro et al. 1995a). A brief review of previous methods (developed before 1992) can be found in Qiu and Xu (1992), and this paper reports a three-dimensional extension of the previous two-dimensional simple adjoint (SA) method for wind retrievals.

The SA method was developed by Qiu and Xu (1992) and then improved and tested with the Phoenix II data (Xu et al. 1994a,b; Xu and Qiu 1995), the Denver microburst data (Xu et al. 1995), and the “bake-off” data in an intercomparison project of single-Doppler velocity retrievals (Shapiro et al. 1995b, 1996). The method is called ‘adjoint’ as the adjoint technique is used to minimize the difference between the observed and model-predicted reflectivities and/or radial winds (Lewis and Derber 1985; Talagrand and Courtier 1987). The method is called ‘simple adjoint’ because the adjoint is applied only to the reflectivity conservation equation and/or the radial-wind momentum equation. The full adjoint method integrates the full model equations and thus requires specifications of the unknown boundary values of cross-beam winds and temperature for the time integration. The SA method integrates only the reflectivity conservation equation and/or the radial-wind momentum equation, so the required boundary conditions can be given by single-Doppler scans. The primary idea of the

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SA method is to retrieve the time-mean winds by using a sequence of radar scans (several time levels) over a relatively short time window. As multiple time levels of radar observations are used by the SA method, the retrieved winds become more accurate and less sensitive to data errors. The full adjoint method also retrieves the pressure and temperature fields, but the SA method does not. This is a limitation of the previous SA method.

Previous tests and applications of the SA method with real Doppler-radar data were limited to two-dimensional wind retrievals over low-altitude horizontal planes. The vertical wind fields were not directly retrieved, but computed afterward from the retrieved horizontal winds by using the mass continuity equation. In this sense, the previous SA method is essentially 'two-dimensional' and may be called 2D-SA for short. In the 2D-SA method, the pressure gradient term and the buoyancy term are treated as an unknown source in the radial-wind momentum equation. Although the time mean of this unknown source can be retrieved as the residual of the equation for the purpose of improving the wind retrievals (Xu et al. 1994b, 1995), the perturbation pressure and buoyancy fields are not treated explicitly in the 2D-SA method.

In this paper, the SA method will be upgraded to make use of the full momentum equations together with the mass continuity equation in three-dimensional space. These equations will be used diagnostically as weak constraints (Sasaki 1970) to avoid specifications of the unknown boundary values of cross-beam winds and temperature which were required by the time integration of these equations in the full adjoint method (Sun and Crook 1994). These weak constraints relate the perturbation pressure and buoyancy to the wind field, so the pressure gradient term and the buoyancy term can be explicitly estimated in the radial-wind momentum equation to improve the wind retrievals. The detailed formulations are presented in the next section. The numerical schemes are described in section 3. The results of numerical experiments with real Doppler-radar data are shown in section 4. The effects of smoothness constraints and sensitivities to weights are examined in section 5. Conclusions follow in section 6.

2. EQUATIONS AND COST FUNCTIONS

The radial-wind momentum equation has the following form:

$$\frac{\partial v_r}{\partial t} + \nabla \cdot (rv_r) = \frac{|v|^2 - v_r^2}{r} + r \cdot (kb - \nabla \eta) + \kappa \nabla^2 v_r, \quad (2.1)$$

where $v_r$ is the radial-component wind, $v$ is the vector velocity in three-dimensional space, $r$ is the unit vector along the radar beam, $r$ is the radial coordinate (distance from the radar), $\nabla$ is the gradient operator in the Cartesian coordinate system with $\nabla \cdot \nabla = \nabla^2$, $b$ is the vertical unit vector, $\kappa (\approx 100 \text{ m}^2 \text{ s}^{-1})$ is a constant coefficient of eddy diffusivity, $\eta$ is the perturbation pressure divided by a reference density, $b = g \theta / \Theta$ is the buoyancy, $\theta$ is the potential-temperature perturbation, and $\Theta$ is the horizontal-mean potential temperature. The reflectivity advection equation has the following form:

$$\frac{\partial \eta}{\partial t} + \nabla \cdot (v \eta) = \kappa \nabla^2 \eta + S, \quad (2.2)$$

where $\eta$ is the (logarithmic) reflectivity in non-dimensional units, and $S$ is the unknown source term (which might be negligible for clear-air reflectivity but not for hydrometeor reflectivity). The time mean of this source term is to be retrieved as in Xu and Qiu (1995). Since both $v_r$ and $\eta$ are observed by a single-Doppler radar, the initial conditions and inflow boundary conditions for Eqs. (2.1) and (2.2) can be given observationally.
In the previous 2D-SA method (Xu et al. 1994b, 1995), the second term on the right-hand side of Eq. (2.1) is retrieved as an unknown source term. In the new method, this source term is expressed explicitly in terms of perturbation pressure and buoyancy, so the perturbation pressure and buoyancy can be related to velocity by

$$\nabla^2 \pi - \mathbf{k} \cdot \nabla b + \nabla \cdot (\mathbf{v} \cdot \nabla \mathbf{v}) = 0. \quad (2.3)$$

Here, Eq. (2.3) is derived from the following vector momentum equation (with the Boussinesq approximation)

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} - \mathbf{k}b + \nabla \pi - \kappa \nabla^2 \mathbf{v} = 0, \quad (2.4)$$

and mass continuity equation

$$\nabla \cdot \mathbf{v} = 0. \quad (2.5)$$

In the derivation of Eq. (2.3), the local time derivative and diffusion terms drop out owing to the mass continuity.

The goal is to estimate \((\mathbf{v}, \pi, b)\) that minimize the following cost function

$$J = \int_0^T \int_{\Omega} \left\{ w_1(\eta - \eta_{\text{ob}})^2 + w_2(v_r - v_{r\text{ob}})^2 + w_3(v \cdot r - v_{r\text{ob}})^2 \\
+ w_4(2.3)^2 + w_5(2.4)^2 + w_6(2.5)^2 \right\} \, dx \, dy \, dz \, dt \quad (2.6)$$

where \(v_r\) and \(\eta\) are 'predicted' over the retrieval time period, \(\tau\), by Eqs. (2.1) and (2.2) with the observed initial and boundary conditions, \((\cdot)_{\text{ob}}\) denotes the observation of \((\cdot)\), and the equation numbers (2.3)–(2.5) refer to the left-hand sides of these equations. In Eq. (2.6), \(v_r\) represents the radial wind predicted by Eq. (2.1) and \(\mathbf{v}\) represents the retrieved vector velocity, so \(J_2\) measures the difference between the predicted and observed radial winds and \(J_3\) measures the difference between the retrieved and observed radial winds. The weights \((w_1, w_2, w_3, w_4, w_5, w_6)\) are data dependent and are given by

$$w_1 = \{(0.1|\delta \eta_{\text{ob}}|)\}^{-2},$$

$$w_2 = \{(0.1|\delta v_{r\text{ob}}|)\}^{-2},$$

$$w_3 = 1.28w_2,$$

$$w_4 = 0.2V^4 A^{-4},$$

$$w_5 = 3.3A^{-2},$$

$$w_6 = 1.3 \times 10^6 \text{s}^2,$$

where \(\{\cdot\}\) represents the averaged value over all the grid points, \(\delta(\cdot)\) denotes the difference between the initial and final values of \((\cdot)\) over the retrieval time period \(\tau\), \(V = 0.5\{(v_{r\text{ob}}(0)) + |v_{r\text{ob}}(\tau)|\}\), and \(A = \{(|\delta v_{r\text{ob}}|)\}/\tau\). Note that \(A\) measures the local acceleration and Lagrangian acceleration, so \(V^2 A^{-1}\) can be used as a length scale to measure the spatial variations of the winds and \(A/(V^2 A^{-1}) = A^2 V^{-2}\) can be used to measure the terms in the pressure equation (2.3). The coefficients in the above weight formulations are obtained empirically (through a large number of numerical experiments) based on considerations similar to those in Xu et al. (1994a,b) and Xu and Qiu (1995). The gradient of \(J_1 + J_2\) with respect to \((\mathbf{v}, \pi, b, S)\) can be derived by using the adjoints of Eqs. (2.1) and (2.2) similarly to Qiu and Xu (1992) and Xu and Qiu (1995), and the gradient of \(J_3 + J_4 + J_5 + J_6\) with respect to \((\mathbf{v}, \pi, b)\) can be derived by using the chain rule for differentiation. The detailed formulations are omitted here.
3. Numerical schemes

Since observational data for $v_t$ and $\eta$ are available on a Cartesian grid, and these gridded data will be used for the tests in the next section, the strong-constraint Eqs. (2.1) and (2.2) are discretized on the same Cartesian grid. At the interior points, the flux-form central finite-difference scheme (second order) is used for the advection terms. The Euler backward scheme is used for the first time step and is followed in all subsequent time steps by the leap-frog scheme for the time integrations of $v_t$ and $\eta$. At the boundary points, $v_t$ and $\eta$ are given by observations. At the lower (surface) boundary, $v_t = 0$ is assumed (consistent with the non-slip boundary condition used for the retrieved velocity) and $\eta$ is set to be the same as in the next level above. The retrieved variables $(v, \pi, b, S)$ are expressed by continuous basis functions over a coarse finite-element mesh on each vertical level. Quadratic bi-spline basis functions are used for the time-mean fields of $(v, \pi, b, S)$, while linear bi-spline basis functions are used for the time tendencies of $(v, \pi, b, S)$ over the retrieval time window, $\tau$. The retrieved fields on the $k$th vertical level have the following forms:

$$v(k) = \tilde{v}_{ijk} B_i^\alpha(x) B_j^\beta(y) + (t - 0.5\tau) (\partial v / \partial t)_{ijk} L_i(x) L_j(y), \quad (3.1)$$

$$\pi(k) = \pi_{ijk} B_i^\alpha(x) B_j^\beta(y) + (t - 0.5\tau) (\partial \pi / \partial t)_{ijk} L_i(x) L_j(y), \quad (3.2)$$

$$b(k) = b_{ijk} B_i^\alpha(x) B_j^\beta(y) + (t - 0.5\tau) (\partial b / \partial t)_{ijk} L_i(x) L_j(y), \quad (3.3)$$

$$S(k) = S_{ijk} B_i^\alpha(x) B_j^\beta(y) + (t - 0.5\tau) (\partial S / \partial t)_{ijk} L_i(x) L_j(y), \quad (3.4)$$

where the summation convention (implied by double indices) is used, $B_i^\alpha(x) B_j^\beta(y)$ denotes the quadratic bi-spline basis function associated with the $ijk$th node and derivatives of order $\alpha$ in $x$ and order $\beta$ in $y$ (where $\alpha = 0, 1$ and $\beta = 0, 1$), and $L_i(x) L_j(y)$ denotes the linear bi-spline basis function associated with the $ijk$th node. These basis functions are constructed as described in the appendix.

The coefficients in Eqs. (3.1)–(3.4) are unknown and need to be retrieved as the minimizers of the cost-function in Eq. (2.6). These coefficients can be denoted by two vectors

$$C_m = [\tilde{v}_{ijk}, \pi_{ijk}, b_{ijk}, S_{ijk}]^T, \quad (3.5)$$

$$C_t = [(\partial v / \partial t)_{ijk}, (\partial \pi / \partial t)_{ijk}, (\partial b / \partial t)_{ijk}, (\partial S / \partial t)_{ijk}]^T, \quad (3.6)$$

where $[\ ]^T$ is the transpose of $[\ ]$ and denotes a vertical column. Note that for each element node there are four quadratic bi-spline basis functions but only one linear bi-spline basis function, so the dimension of $C = [C_m, C_t]^T$ is \((6 \times 4) + 6\) by \(m_x m_y M_z = 30 m_x m_y M_z\) where $m_x$ and $m_y$ are the numbers of element nodes along the $x$-direction and $y$-direction, respectively, and $M_z$ is the number of vertical levels. When the continuous fields expressed by $C_m$ or $C_t$ are discretized on the same fine grid where Eqs. (2.1) and (2.2) are discretized, the discretized fields contain $M_x M_y M_z$ grid points, where $M_x$ and $M_y$ are the numbers of grid points in the $x$- and $y$-directions, respectively. These discretized fields can be denoted by vector $G_m$ (or $G_t$), in association with $C_m$ (or $C_t$). The transformation from $C = [C_m, C_t]^T$ to $G = [G_m, G_t]^T$ can be denoted by $G = FC$. By substituting $G = FC$ into the discretized equations (2.1)–(2.2) and $J_1 + J_2$ (see Eq. (2.6)), the gradient of $J_1 + J_2$ with respect to $C$ is derived from the adjoint of the discretized Eqs. (2.1)–(2.2). The numerical code is developed and checked to ensure the conjugate property between the discretized forward operator and adjoint operator.
The gradient formulation for the remaining part of the cost function can be derived by substituting Eqs. (3.1)–(3.4) into \( J_3 + J_4 + J_5 + J_6 \) and differentiating the resulting formulation with respect to \( C \). The details are omitted here.

The standard conjugate gradient algorithm UMCGG* is used together with the discretized cost function \( J = J_1 + J_2 + J_3 + J_4 + J_5 + J_6 \) (see Eq. (2.6)) and its gradient to search for the minimum of \( J \) in the space of \( C \). The components of \( C \) are grouped in terms of control variables and their derivatives, and the grouped variables are scaled properly so that the gradient components can have about the same non-dimensional magnitudes with respect to different groups of variables. This treatment is similar to that in the previous 2D-SA method (see section 2(b) of Xu and Qiu (1995)), and the purpose is to reduce the ellipticity of the cost function and thus to improve the convergence of the descent algorithm. In this paper, all the iterations start with zero first-guess fields.

Our previous studies (Xu et al. 1994a,b; Yang and Xu 1996) suggest that when the wind fields are retrieved on the same high-resolution spatial grid as the radar data, the retrieved fields need to be constrained by a spatial smoother to filter short-wave noise. However, when the retrieved fields are smoothed, the original grid resolution may become excessive and unnecessary. As shown in Qiu and Xu (1994), the excessive spatial resolution can be avoided when the retrieved field is expressed by a spectral expansion. Because of this, and its effectiveness in filtering short-wave error noise, the spectral expansion can improve the retrievals. In this paper, the retrieved fields are expressed by expansions of quadratic bi-spline basis functions on a coarse finite-element mesh, so the dimension of the retrieved variables can be reduced. The finite-element mesh is \( 4 \times 4 \) times coarser than the data grid mesh in the horizontal. Note that the dimension of \( C \) is \( 30m_xm_yM_z = (30/16)M_xM_yM_z \). The dimension of \( G \) is \( 12M_xM_yM_z \) which is \( 12 \times 16/30 \approx 6 \) times as large as the dimension of \( C \). When the bi-spline representation is not used, the number of unknowns (\( = 12M_xM_yM_z \)) is found to be too large to be retrieved. In this case, to reduce the number of unknowns, \( G_t \) has to be set to zero and not retrieved. Thus, in this paper, all the grid-point retrievals (without using the bi-spline representation in Eqs. (3.1)–(3.4)) are obtained by setting \( G_t = 0 \) (that is, neglecting the time tendencies of the retrieved \( (v, \pi, b, S) \)).

4. Test results with Phoenix II data

(a) Data description

The method is tested with the same radar data previously used by Sun and Crook (1994), hereafter referred to as SC94. The data were collected for a gust front observed during the Phoenix II field experiment. This experiment was conducted in 1984 on the High Plains of eastern Colorado to study the convective boundary layer. The gust front occurred in the afternoon of 19 June 1984. This gust front, caused by evaporative cooling from an earlier thunderstorm, propagated north-westward at a speed of about 7.2 m s\(^{-1}\) through the dual-Doppler analysis area (see Fig. 1 of SC94). Soundings taken before and after the gust-front passage showed that the atmospheric boundary layer extended 1.2 km above ground level where it was capped by a weak temperature inversion. The air behind the front was about 3 degC cooler than that ahead of the front (Mueller and Carbone 1987).

Radial winds and/or reflectivity measured by the two X-band radars (National Oceanic and Atmospheric Administration, C and D) are used to test our method.

As described in SC94, the data were preprocessed (with folded velocities corrected), interpolated onto Cartesian coordinates and "shifted" to a common time level for each volume scan. The gridded data used by SC94 in their assimilation experiments covered an $8 \times 8 \times 4 \text{ km}^3$ domain with grid spacings of 400 m in the horizontal and 200 m in the vertical. The same gridded data are used here for our tests, but the domain height is reduced from 4 km to 2.8 km in order to deal with the following two problems: (i) the original data coverage is poor above the boundary layer and reliable gridded data are available only up to a height of 2.8 km; (ii) the current SA method needs observed initial and boundary conditions (including the upper-boundary condition) for the integrations of Eqs. (2.1) and (2.2). In the full adjoint method of SC94, the upper-boundary condition was assumed to be a rigid lid, the lateral-boundary conditions were given by using prior information (from dual-Doppler observations and/or soundings), and the initial fields were treated as unknowns to be retrieved. The computational domain had to be sufficiently deep (at 4 km) to reduce the influence of the upper rigid lid in SC94, although there was little data above 2.8 km.

(b) Retrieved wind fields

The selected data cover four time levels (1–4), 110 s apart, over the period of 2231:20–2236:50 UTC, 19 June 1984. The retrieval time period is chosen to be $\tau = 110$ s, covering two consecutive time levels, so the selected data cover three time windows between time levels 1–2, 2–3 and 3–4. Over each time window, three types of retrievals are performed by using the radial-wind data only (with $w_1 = 0$), reflectivity data only (with $u_2 = 0$), and both radial-wind and reflectivity data. When the wind fields are retrieved directly on the high-resolution data grid without using the bi-spline representation in Eqs. (3.1)–(3.4), the retrieval from radar C (or radar D) using the radial-wind and reflectivity data requires 1200 (or 3000) iterations for the descent algorithm to converge satisfactorily (measured by the root-mean-square (r.m.s.) difference between the retrieval and dual-Doppler observation). By using the bi-spline representation in Eqs. (3.1)–(3.4), the retrieval from radar C (or radar D) requires only 500 (or 2000) iterations for the descent algorithm to converge satisfactorily. (It is noted that the cost function decreases rapidly (by more than 95%) before the number of iterations reaches 100. The slow convergence after 100 iterations may imply the strong ellipticity of the cost function in the sub-space expanded by the pressure and temperature variables that are nonlinearly and remotely linked to the retrieved wind through the weak constraints.)
Figure 1. Retrieved wind fields in experiment 1.03 (see text): (a) horizontal wind at $z = 0.6$ km, (b) vertical velocity at $z = 0.6$ km, (c) horizontal wind at $z = 2$ km, and (d) vertical velocity at $z = 2$ km. Contours for the vertical velocity are every 0.5 m s$^{-1}$.

The domain-averaged absolute differences (DD), relative DD (RDD) and spatial correlation coefficients (SCC) between the retrieved and dual-Doppler analysed horizontal winds are computed by using the following formulations:

$$\text{DD} = \langle |v_1 - v_1^a| \rangle, \quad (4.1)$$
$$\text{RDD} = \text{DD} \langle |v_1^a| \rangle^{-1}, \quad (4.2)$$
$$\text{SCC} = \langle (v_1 - v_1^a)(v_1^a)' \rangle \{ \langle (v_1^a)' \rangle^2 \}^{-1/2}. \quad (4.3)$$

where $v_1$ and $v_1^a$ are the retrieved and dual-Doppler analysed tangential velocity components in the horizontal, respectively, $\langle \rangle$ denotes the domain-averaging operator over the retrieval volume, $\langle \rangle_m$ denotes the time-averaging operator over period $\tau$, and $(\cdot)' = (\cdot) - \langle \rangle$. The results are summarized for ten experiments in Table 1, where the
Figure 2. As in Fig. 1 but for dual-Doppler analysed wind fields.

statistics for the retrieved tangential winds, with and without using the bi-spline representation, are listed under the names of 'bi-spline model' and 'grid-point model', respectively. Experiments 1.01–1.03 and 1.06–1.08 are retrievals over the first time window (between time levels 1 and 2), and these results show that the third-type retrieval (using both radial-wind and reflectivity data) is slightly better than the first type (using radial wind only) and the latter is better than the second type (using reflectivity data only). This is also true for the retrievals over the second and third time windows, although only the third-type retrievals are listed for the latter two time windows in Table 1. Thus, a properly combined use of radial-wind and reflectivity data can improve the retrieval, and this agrees with the results obtained by Xu and Qiu (1995).

The retrieved wind fields obtained with the bi-spline representation in experiment 1.03 (from radar C, by using both the radial-wind and reflectivity data over time levels 1 and 2) are shown in Figs. 1(a)–(d) for two vertical levels. Comparing these fields with
the corresponding dual-Doppler analysed fields in Figs. 2(a)–(d), we can see that the retrieved horizontal winds match the dual-Doppler winds very well at both vertical levels. At $z = 0.6$ km, the retrieved vertical velocity shows a strong updraught band along the gust front, and this structure agrees well with the dual-Doppler analysis. This updraught band becomes relatively weak as it extends to the upper level ($z = 2$ km) in the retrieved velocity field. This general feature also agrees with the dual-Doppler analysis.

The dual-Doppler analysed vertical velocity contains enhanced irregular small-scale structures at the upper level (Fig. 2(d)). These small-scale structures are not seen in the retrieval (Fig. 1(d)) and the reasons could be twofold. First, since quadratic bi-spline basis functions are used with a coarse finite-element mesh to filter short-wave noise in the retrievals, small-scale structures cannot be well resolved by the retrievals. Second, the dual-Doppler analysis of vertical velocity is calculated by vertically integrating the mass continuity equation (2.5) from the surface level (where the vertical velocity is zero) with the horizontal wind divergence estimated from observations. Small-scale errors could be amplified in the divergence computation and accumulated in the vertical integration. This may explain why the dual-Doppler analysed vertical-velocity field becomes increasingly noisy at the upper levels (as shown by Figs. 2(b) and (d)) and why these noisy small-scale structures are not consistent with time (not shown).

(c) Retrieved thermodynamic fields

The perturbation potential temperature retrieved with the bi-spline representation in experiment 1.03 is plotted in Fig. 3 at $z = 0.6$ km. As shown, the air behind the gust front is cooler than air ahead of the front, but the temperature difference (about 1 degC) seems to be smaller than that observed (about 3 degC near the surface according to SC94). The retrieved perturbation pressure (not shown) is relatively high along the gust front but relatively low away from the gust front. The high pressure along the gust front may be explained by the Bernoulli effect associated with the flow deceleration immediately ahead of the front, but the relatively low pressure away from and behind the gust front is not consistent with the far-field hydrostatics of density currents (Benjamin 1968; Xu 1992). Thus, the retrieved perturbation pressure and buoyancy may not be
TABLE 2. AS IN TABLE 1 BUT WITH THE SMOOTHNESS CONSTRAINTS DEFINED BY EQ. (5.1)

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<th>RDD</th>
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<td>1.325</td>
<td>0.312</td>
<td>0.927</td>
<td>3600</td>
<td>1.472</td>
<td>0.346</td>
<td>0.909</td>
</tr>
<tr>
<td>2.10</td>
<td>D</td>
<td>$V_r$ &amp; $\eta$</td>
<td>3–4</td>
<td>1900</td>
<td>1.588</td>
<td>0.344</td>
<td>0.921</td>
<td>7700</td>
<td>1.858</td>
<td>0.403</td>
<td>0.899</td>
</tr>
</tbody>
</table>

accurate, although their errors cannot be quantified here due to lack of high-resolution observations of pressure and temperature.

As explained earlier, the second term on the right-hand side of Eq. (2.1) was treated as an unknown source term in the previous 2D-SA method, but now is expressed explicitly in terms of perturbation pressure and buoyancy. Implicitly, however, this explicit term absorbs the unknown residual error of Eq. (2.1). Similarly, the perturbation pressure and buoyancy terms in Eqs. (2.3) and (2.4) also absorb the residual errors of Eqs. (2.3) and (2.4). Thus, with the current three-dimensional simple adjoint (3D-SA) method, the explicit term of perturbation pressure and buoyancy in Eq. (2.1) is recovered mainly for the purpose of improving the wind retrieval. This explicit term can provide a better estimate of the source term (including the unknown equation error) which was treated as unknown in the previous 2D-SA method, although the retrieved perturbation pressure and buoyancy may not be accurate.

5. THE EFFECTS OF SMOOTHNESS CONSTRAINTS AND SENSITIVITIES TO WEIGHTS

(a) Experiments with smoothness constraints

In this section, additional experiments are performed to examine whether, and how much, the retrievals can be improved by adding smoothness constraints to the cost function. Since $2\Delta x$ noise can be effectively filtered by five-point averaging in the horizontal, only this type of smoothing is examined in this paper. Furthermore, since our results indicate that applying the smoothness constraint to $(x, b)$ in the cost function does not improve the retrievals in general, we will only present the results obtained by applying the smoothness constraint to $(v, s)$ in the cost function. In this case, the cost function $J$ defined in Eq. (2.6) is upgraded as follows:

$$J_s = J + \int_0^\infty \int_\Omega \left\{ w_7 |\Delta_H v|^2 + w_8 (\Delta_H s)^2 \right\} \, dx \, dy \, dz \, dt$$
$$= J + J_7 + J_8,$$

where $\Delta_H$ represents five-point averaging on the horizontal grid, and the optimal values (in terms of orders of magnitudes) for the two weights are found to be $w_7 = 50$ m$^{-2}$ s$^2$ and $w_8 = 5 \times 10^5$ s$^2$.

The results of the above experiments are summarized in Table 2. In comparison with the results in Table 1, the improvements in Table 2 are seen as follows:
TABLE 3. AS IN TABLE 2 BUT ONLY FOR THE THIRD-TYPE RETRIEVALS

<table>
<thead>
<tr>
<th>Model</th>
<th>Radar</th>
<th>Time levels</th>
<th>$S = 0$</th>
<th>$S$ retrieved</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$n$</td>
<td>DD</td>
</tr>
<tr>
<td>C</td>
<td>1–2</td>
<td>300</td>
<td>0.865</td>
<td>0.276</td>
</tr>
<tr>
<td>C</td>
<td>2–3</td>
<td>300</td>
<td>1.086</td>
<td>0.365</td>
</tr>
<tr>
<td>Bi-</td>
<td></td>
<td>3–4</td>
<td>600</td>
<td>1.260</td>
</tr>
<tr>
<td>spline</td>
<td>D</td>
<td>1–2</td>
<td>600</td>
<td>1.356</td>
</tr>
<tr>
<td></td>
<td>D</td>
<td>2–3</td>
<td>600</td>
<td>1.624</td>
</tr>
<tr>
<td></td>
<td>D</td>
<td>3–4</td>
<td>700</td>
<td>1.664</td>
</tr>
</tbody>
</table>

Using both radial-wind and reflectivity data in comparison with those with $S = 0$. See text for further explanation.

(i) The grid-point retrievals are greatly improved but they are still not as accurate as the bi-spline retrievals.

(ii) The bi-spline retrievals are also improved (except for experiment 2.01), but the improvements are not as great as for the grid-point retrievals and the vertical-motion fields become too weak and too smooth (not shown) compared with the dual-Doppler analyses.

(iii) The retrievals from radar D are improved more than the retrievals from radar C. This seems to suggest that radar D data are noisier than radar C data.

(iv) The retrievals from the reflectivity data only are improved more than the retrievals from radial-wind data only. This seems to suggest that the reflectivity data are noisier than the radial-wind data.

(v) Relatively poor retrievals in Table 1 gain relatively large improvements in Table 2, so the statistics are more uniformly distributed among the 10 experiments in Table 2 than in Table 1. This implies that the robustness of the method can be improved by the smoothness constraints, especially when the bi-spline representation is not used. This is consistent with the theoretical results of Yang and Xu (1996).

(b) Experiments without retrieving the reflectivity source

The results in Table 2 for the third-type retrievals (using both radial-wind and reflectivity data) are also listed in Table 3 in comparison with those without retrieving the reflectivity source ($S = 0$). As shown by the comparison, when $S$ is set to zero, the retrievals become slightly worse but the number of iterations are reduced significantly. The improved convergence can be related to the reduced number of unknowns (from $30m_xm_yM_z$ to $25m_xm_yM_z$). Thus, neglecting the reflectivity source term can increase the computational efficiency for practical applications, while the loss of accuracy is insignificant (especially for the bi-spline retrievals).

(c) Sensitivities to weights

Experiments are performed to examine the sensitivities of the retrievals to variations (from 0.1 to 10 times) of the weights ($w_4, w_5, w_6, w_7$) specified in the cost function (see Eqs. (2.6) and (5.1)). The results are summarized in Table 4, where the reference experiment is listed on the first line, which combines line two (radar C with $S = 0$) and
TABLE 4. Sensitivities of the retrieved tangential winds to the four weights ($w_4$, $w_5$, $w_6$, $w_7$)

| Constraint | Weight | Radar C | | | | Radar D | | | |
|------------|--------|--------|--------|--------|--------|--------|--------|--------|
| All        | ×1     | 300    | 1.086 | 0.365 | 0.748 | 600    | 1.624 | 0.382 | 0.872 |
| Continuity | /10    |        |       |       |       |        |       |       |       |
| $w_6$      | ×10    | 300    | 1.081 | 0.364 | 0.745 | 500    | 1.701 | 0.400 | 0.868 |
| Pressure   | /10    |        |       |       |       |        |       |       |       |
| $w_5$      | ×10    | 200    | 1.108 | 0.372 | 0.737 | 600    | 1.509 | 0.355 | 0.888 |
| Momentum   | /10    |        |       |       |       |        |       |       |       |
| $w_4$      | ×10    | 400    | 1.146 | 0.386 | 0.723 | 500    | 1.565 | 0.368 | 0.879 |
| Smoothing  | /10    |        |       |       |       |        |       |       |       |
| $w_7$      | ×10    | 400    | 1.139 | 0.383 | 0.740 | 500    | 1.682 | 0.396 | 0.873 |

The reference experiment is listed on the first line which combines line two (Radar C with $S = 0$) and line five (Radar D with $S = 0$) from Table 3. See text for further explanation.

TABLE 5. As in Table 4 but obtained by multiplying $w_4$, $w_5$, $w_6$ and $w_7$ by either 0 or 1 (indicated by a blank space) in various combinations.

<table>
<thead>
<tr>
<th>Exp. no.</th>
<th>Weights</th>
<th>Radar C</th>
<th></th>
<th></th>
<th></th>
<th>Radar D</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>5.01</td>
<td>0</td>
<td>300</td>
<td>1.086</td>
<td>0.365</td>
<td>0.748</td>
<td>600</td>
<td>1.624</td>
<td>0.382</td>
<td>0.872</td>
</tr>
<tr>
<td>5.02</td>
<td>0</td>
<td>300</td>
<td>1.082</td>
<td>0.364</td>
<td>0.745</td>
<td>500</td>
<td>1.731</td>
<td>0.407</td>
<td>0.865</td>
</tr>
<tr>
<td>5.03</td>
<td>0</td>
<td>200</td>
<td>1.105</td>
<td>0.372</td>
<td>0.739</td>
<td>400</td>
<td>1.644</td>
<td>0.387</td>
<td>0.876</td>
</tr>
<tr>
<td>5.04</td>
<td>0</td>
<td>200</td>
<td>1.166</td>
<td>0.392</td>
<td>0.715</td>
<td>500</td>
<td>1.600</td>
<td>0.376</td>
<td>0.878</td>
</tr>
<tr>
<td>5.05</td>
<td>0</td>
<td>400</td>
<td>1.190</td>
<td>0.400</td>
<td>0.735</td>
<td>700</td>
<td>1.740</td>
<td>0.409</td>
<td>0.885</td>
</tr>
<tr>
<td>5.06</td>
<td>0</td>
<td>200</td>
<td>1.097</td>
<td>0.369</td>
<td>0.737</td>
<td>500</td>
<td>1.811</td>
<td>0.426</td>
<td>0.849</td>
</tr>
<tr>
<td>5.07</td>
<td>0</td>
<td>300</td>
<td>1.173</td>
<td>0.394</td>
<td>0.707</td>
<td>500</td>
<td>1.764</td>
<td>0.415</td>
<td>0.862</td>
</tr>
<tr>
<td>5.08</td>
<td>0</td>
<td>200</td>
<td>1.226</td>
<td>0.412</td>
<td>0.683</td>
<td>400</td>
<td>1.728</td>
<td>0.407</td>
<td>0.853</td>
</tr>
<tr>
<td>5.09</td>
<td>0</td>
<td>200</td>
<td>1.211</td>
<td>0.407</td>
<td>0.687</td>
<td>500</td>
<td>1.895</td>
<td>0.446</td>
<td>0.841</td>
</tr>
<tr>
<td>5.10</td>
<td>0</td>
<td>300</td>
<td>1.269</td>
<td>0.427</td>
<td>0.697</td>
<td>700</td>
<td>2.144</td>
<td>0.504</td>
<td>0.798</td>
</tr>
<tr>
<td>5.11</td>
<td>0</td>
<td>300</td>
<td>1.483</td>
<td>0.499</td>
<td>0.643</td>
<td>400</td>
<td>1.917</td>
<td>0.451</td>
<td>0.815</td>
</tr>
<tr>
<td>5.12</td>
<td>0</td>
<td>500</td>
<td>1.536</td>
<td>0.517</td>
<td>0.626</td>
<td>500</td>
<td>2.088</td>
<td>0.491</td>
<td>0.792</td>
</tr>
<tr>
<td>5.13</td>
<td>0</td>
<td>300</td>
<td>1.688</td>
<td>0.568</td>
<td>0.592</td>
<td>500</td>
<td>2.253</td>
<td>0.530</td>
<td>0.744</td>
</tr>
</tbody>
</table>

The reference experiment 5.01 is the same as in Table 4.

line five (Radar D with $S = 0$) in Table 3. As shown, the variations in the statistics are within 10% when $w_4$, $w_5$, or $w_6$ are increased or decreased by 10 times. The variations in the statistics are relatively large but do not exceed 15% when $w_7$ (the weight for the smoothness constraint on $v$) is increased or decreased by 10 times. Clearly, the retrievals are not sensitive to the above four weights and especially not sensitive to $w_5$ (the weight for the pressure equation).

Table 5 lists the results of experiments with $w_4$, $w_5$, $w_6$ and/or $w_7$ set to zero (to eliminate some or all of the four weak constraints) in various combinations. The reference experiment 5.01 is the same as in Table 4. By comparing experiments 5.02–5.05 with 5.01 and comparing experiments 5.09–5.12 with 5.13, we can see that the retrievals from radar C are influenced most by the smoothness constraint and then by the vector-momentum-equation constraint, while the retrievals from radar D are influenced most by the smoothness constraint and then by the continuity-equation constraint. When the smoothness constraint is not used (compare experiments 5.06–5.08 with 5.09),
the retrievals from radar C are influenced most by the vector-momentum-equation constraint, while the retrievals from radar D are influenced most by the continuity-equation constraint. Thus, among the four weak constraints, the smoothness constraint is the strongest and the pressure-equation constraint is the weakest in terms of their relative impacts on the retrievals.

6. CONCLUSIONS

In this paper, the previous 2D-SA method is further upgraded to a 3D-SA method. This 3D-SA method uses the full momentum equations together with the mass continuity equation as weak constraints in addition to the strong constraints of the radial-wind and reflectivity equations used by the previous 2D-SA method. The method is tested with Doppler-radar data collected during the Phoenix II field experiment. The retrieved winds are very close to the dual-Doppler observed winds and the r.m.s. difference is about 1.0 m s\(^{-1}\). A series of experiments is performed to examine the sensitivities of the retrievals to selections of weights and different numerical schemes.

The weak constraints used in the 3D-SA method relate the perturbation pressure and buoyancy to the wind field, so the unknown source term in the previous 2D-SA method can be explicitly estimated in terms of perturbation pressure and buoyancy. This improves the wind retrievals (in comparison with the retrievals obtained by the previous 2D-SA method).

Quadratic bi-spline basis functions are used to express the retrieved fields on a coarse finite-element mesh to filter short-wave error noise, so no additional spatial smoother is needed in the cost function. However, if the retrievals are directly expressed on the same Cartesian grid as the data without using the bi-spline representations (see Eqs. (3.1)–(3.4)), then the convergence of the iterative descent algorithm will be slowed by about a factor of two and the differences (DD) between the retrieved and dual-Doppler observed winds will be increased by about 1.5 times (see Table 1). The bi-spline representation not only reduces the number of unknowns to be retrieved but also improves the accuracy of the retrievals.

Applying smoothness constraints to the retrieved vector wind and reflectivity source can improve the accuracy of the retrievals as well as the robustness of the method, especially when the bi-spline representation is not used (see Table 2). Among the four weak constraints, the smoothness constraint is the strongest and the pressure-equation constraint is the weakest in terms of the relative impacts on the retrievals (see Table 5).

The retrievals are found to be insensitive to the weights specified for the four weak constraints in the cost function (see Table 4). Retrieving the reflectivity source term can slightly improve the accuracy of the retrievals but requires a relatively large number of iterations (see Table 3). Thus, neglecting the reflectivity source term can improve the computational efficiency with little, or almost no, loss of accuracy, especially when the bi-spline representation is used.

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Appendix

Construction of B-spline basis functions

The one-dimensional basis function \( B_i^{\alpha}(x) \) (with fixed \( i \) and \( \alpha \)) and its derivatives of order \( \alpha' = 0, 1 \) at the nodes \( x = x_i' \) (\( i' = 1, 2 \)) satisfy

\[
\partial^{\alpha'} B_i^{\alpha}(x_i') = \delta_{i'i'} \delta_{\alpha\alpha'},
\]
where \( \delta_{i'i'} \) and \( \delta_{\alpha\alpha'} \) are Kronecker deltas. To construct \( B_i^{\alpha}(x) \) over a line element \( \Delta x = x_1 - x_2 \), we divide \( \Delta x \) equally into two segments and thus obtain six constraints for \( B_i^{\alpha}(x) \): four of them are given by Eq. (A.1) at the two nodes (\( x = x_1 \) and \( x = x_2 \)) and the remaining two constraints are the continuity of \( \partial^{\alpha'} B_i^{\alpha}(x) \) (\( \alpha' = 0, 1 \)) at the internal nodes \( x = x_1 + 0.5\Delta x \) between the segments. Note that each quadratic polynomial contains three parameters over each interval of \( \Delta x/2 \) and the total number of parameters is \( 3 \times 2 = 6 \), so \( B_i^{\alpha}(x) \) can be determined uniquely by the Hermite interpolation over \( \Delta x \). The situation for \( B_j^{\beta}(y) \) is similar. The basis function \( L_i(x) \) (with fixed \( i \)) at the nodes \( x = x_i' \) (\( i' = 1, 2 \)) satisfies

\[
L_i(x_i') = \delta_{i'i'}.
\]
By using Eq. (A.2), it is straightforward to construct the linear basis function \( L_i(x_i') \) over a line element \( \Delta x = x_1 - x_2 \). The situation for \( L_j(y) \) is similar.

References

Rinehart, R. B. 1979 'Internal storm motions from a single non-Doppler weather radar'. NCAR/TN-146+STR


