Parametrization of solar radiation in inhomogeneous stratocumulus: Albedo bias

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SUMMARY

Low-level cloud fields with typical albedos between 50% and 70% contribute significantly to short-wave cloud radiative forcing. Consequently, the earth's radiative budget is sensitive to extended stratocumulus which typically occurs in sub-tropical subsidence regions over the oceans. It is therefore vital to represent the cloud albedo of such cloud fields accurately in climate models. Nevertheless, cloud fields are modelled as homogeneous plane-parallel clouds, which leads to an overestimation of the albedo by up to 15% compared with real inhomogeneous clouds, both types of cloud fields having the same mean cloud optical depth. This so-called plane-parallel cloud albedo bias has given rise to a number of studies that present methods for reducing the albedo bias for stratiform clouds in climate models.

Using Monte Carlo simulations and the Independent Pixel Approximation (IPA) the inhomogeneous cloud albedo is investigated with the aim of deriving a parametrization of the albedo bias. The new approach used here is that the variance of the cloud optical depth is parametrized as a function of grid size and mean cloud liquid-water content. The variance of the cloud optical depth is derived from turbulence processes. For the determination of the albedo bias an analytical solution is presented which is based on a simplified distribution function of the cloud optical depth and the IPA method. When used in conjunction with the parametrized variance of the cloud optical depth, a box function, which replaces the distribution function, produces comparable results to Monte Carlo simulations within an acceptable accuracy of a few per cent.

KEYWORDS: Independent Pixel Approximation Inhomogeneous clouds Monte Carlo model

1. INTRODUCTION

Various methods can be used to calculate radiative transfer in plane-parallel cloud fields. These methods, however, do not take into account the important effect that horizontal cloud inhomogeneities have on cloud albedo. An example, showing stratocumulus with cloud inhomogeneities on scales ranging from 1 km to 250 km, is given in Fig. 1. Kobayashi (1991), Barker (1992), Cahalan et al. (1994a), among others, have shown that cloud albedo is reduced considerably by cloud inhomogeneities compared with the albedo of plane-parallel cloud fields, even if in both conditions the mean cloud optical depth, \( \bar{\tau} \), is the same. The so-called plane-parallel homogeneous albedo bias can be written as

\[
\Delta A \equiv A(\bar{\tau}) - A(\tau)
\]

where \( A(\bar{\tau}) \) is the albedo of the plane-parallel cloud and \( A(\tau) \) is the horizontally averaged albedo of the inhomogeneous cloud.

Climatological and meteorological models, referred to as general-circulation models (GCMs), use plane-parallel cloud fields to calculate cloud albedo on the basis of cloud liquid-water content, (sometimes) effective droplet radius, and cloud fraction. Cloud liquid-water content and effective droplet radius determine \( \tau \), and cloud fraction defines the part of the grid cell covered by the plane-parallel cloud. It has been shown in many studies (see below) that neglecting cloud inhomogeneities in GCMs generates unacceptably high albedo at the top of the atmosphere, mainly because of the (unrealistic) plane-parallel cloud representation of extended stratiform cloud fields over the oceans. In order to include the important effect of cloud inhomogeneity on

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cloud albedo in GCMs, the authors of several studies have proposed subgrid-scale parametrizations for the albedo bias. Cahalan et al. (1994a) discussed the albedo bias thoroughly, and presented a successful 'effective optical thickness' approach based on a log-normal distribution function of the cloud optical depth. For that purpose Cahalan et al. (1994a) used liquid-water-path observations and a bounded cascade cloud model which provided information on statistical moments of the cloud optical depth distribution. Tiedke (1996) used results obtained by Cahalan et al. (1994a) to extend the parametrization to convective cloud types. Barker (1996) presented a gamma–IPA (IPA = Independent Pixel Approximation) to calculate mean solar radiative fluxes for inhomogeneous marine boundary-layer cloud fields. His analysis was based on Landsat images, the albedo bias of which was discussed by Barker et al. (1996). On the assumption of gamma distributed cloud optical depth fields, Oreopoulos and Davies (1998a,b), and Oreopoulos and Barker (1999) used more satellite observations and mathematics to improve the subgrid-scale parametrization.

The parametrization proposed in the present investigation seeks to elaborate the physical processes which control the cloud optical depth variability. The emphasis is put on two physical cloud processes:
• turbulence,
• macrophysical and microphysical properties.

With these processes one can derive a relation between the cloud optical depth variability and the basic parameters available in GCMs. The turbulence characteristics and cloud macrophysical and microphysical properties were determined from stratocumulus observations obtained during the Atlantic Stratocumulus Transition EXperiment (ASTEX) field campaign (FIRE 1992) and from the Broken and Inhomogeneous Cloud ExPeriment (BICEP) (Francis 1998). Cloud physical and optical properties based on ASTEX observations are described by Hignett and Taylor (1996) and Los and Duynkerke (2000). For the purpose of the present analysis, results obtained by Los and Duynkerke (2000) are used to represent realistic inhomogeneous marine stratocumulus.

Considine and Curry (1996) presented a related study in which they derived the cloud droplet distribution of stratocumulus from turbulent kinetic energy and from a parameter defining the liquid water lapse rate per cloud pixel. However, the statistical model used in their study is formulated to produce horizontally averaged statistics only and consequently does not give the cloud optical depth variability.

For the purpose of the present subgrid-scale parametrization, the inhomogeneous cloud albedo is obtained according to the widely used IPA (Cahalan et al. 1994a,b; Marshall et al. 1995). The distribution function of the cloud optical depth is a conceptually simple box function which permits the derivation of an analytical solution for the IPA. In contrast to previous studies which parametrized the distribution function of the cloud optical depth itself, this study presents a parametrization of the cloud optical depth variability which defines the width of the distribution function, i.e. the box function. Hence, the analytical solution yields the inhomogeneous cloud albedo based on the variability of the cloud optical depth, \( \tau^2 \). For the parametrization of \( \tau^2 \), only the grid size, \( \lambda_c \), and the mean cloud liquid-water content, \( \overline{q_l} \), are required. The ‘flow diagram’ shown below traces the path of the parametrization. The processes and basic parameters required to derive a function for the cloud optical depth variability, which finally yields the cloud albedo, are described below.

\[
\lambda_c \xrightarrow{\text{turbulence}} \overline{q_l}^2 \xrightarrow{\text{macrophysics and microphysics}} \overline{\tau^2} \xrightarrow{\text{radiative transfer}} \Delta A.
\]

**Turbulence.** \( \lambda_c \), and other turbulence boundary-layer parameters determine the cloud liquid-water variability, \( \overline{q_l}^2 \). Turbulence parameters are obtained from stratocumulus observations. To extend the validity of the turbulence boundary-layer parameters (theoretically defined for the inertial subrange only) to \( \lambda_c \), we assume that turbulence characteristics can be extended to the mesoscale range.

**Macrophysics and microphysics.** \( \overline{q_l}^2 \) and other macrophysical and microphysical parameters yield the variability of the cloud optical depth, \( \overline{\tau^2} \). The relationship between \( \overline{\tau^2} \) and \( \overline{q_l}^2 \), and the determination of microphysical parameters, are based on stratocumulus observations (Los and Duynkerke 2000).

**Radiative transfer.** Finally, the IPA method is chosen to calculate \( \Delta A \) of the inhomogeneous cloud field. \( \overline{\tau^2} \) defines the width of the box function in the IPA. Use of the box function lets us derive an analytical solution for the integral in the IPA.

The paper is constructed as follows. Section 2 provides some background on the IPA and introduces two approximations based on the IPA for calculating \( \Delta A \). In the first approximation the albedo function in the IPA is replaced by its Taylor
expansion. The second approximation uses a simplified distribution function of the cloud optical depth field which allows the derivation of an analytical solution of the IPA. In section 3 the bounded cascade cloud model and in situ observations in stratocumulus are presented. These provide the cloud inhomogeneities to be used in the present investigation. Thereafter, in section 4, we present results for the inhomogeneous cloud albedo and for the albedo bias, obtained with the IPA, the approximate Taylor–IPA, and the simplified distribution function. Section 5 introduces the parametrization of the cloud optical depth variability, starting with the variability of the cloud liquid-water content. Thereafter we present the relationship between variances of liquid-water content and cloud optical depth. Next, the impact of conservative scattering, used to obtain the analytical solution with the simplified distribution function, is discussed, and the section ends with the summary of the parametrization. Section 6 contains the summary and conclusions.

2. Theory and Approximation of the Albedo Bias

(a) Independent Pixel Approximation (IPA)

The albedo bias $\Delta A$ of inhomogeneous stratocumulus can be efficiently and accurately determined with the IPA, presented by, for example, Cahalan et al. (1994a). A characteristic constraint of the IPA is that it neglects horizontal photon transport between the pixels. However, this constraint does not restrict the applicability of the IPA as long as horizontal dimensions of the cloud inhomogeneities are well above the so-called radiative smoothing scale $\eta_{\text{rad}}$ which depends on the extinction parameter and the geometrical cloud depth (Marshak et al. 1995). While on horizontal scales larger than $\eta_{\text{rad}}$ the fluctuations of cloud albedo and cloud optical depth are correlated, meaning that the IPA is a valid method for cloud albedo determination (Cahalan et al. 1994a), the similarity between statistical properties of cloud albedo and cloud optical depth disappear on horizontal scales below $\eta_{\text{rad}}$. Following Marshak et al. (1995) $\eta_{\text{rad}}$ is about 100 m for cloud types used in the present analysis. In the present study, cloud inhomogeneities are well above the radiative smoothing scale, $\eta$. According to the IPA, the average cloud albedo is defined as

$$A(\tau) = \int_0^\infty \text{PDF}(\tau) A(\tau) \, d\tau$$

where $A(\tau)$ is the plane-parallel cloud albedo and $\text{PDF}(\tau)$ is the (normalized) probability distribution function obtained either from cloud models or from observations. The IPA is another way of saying that the mean cloud albedo does not depend on the sequence of the inhomogeneities.

Figure 2 shows the modelled $A(\tau)$, and $\text{PDF}(\tau)$ obtained from cloud microphysical observations during run R73 of flight RF06 in the ASTEX campaign. The cloud optical depth, $\tau$, is calculated with Eq. (11), i.e. $\tau$ is determined with the variable cloud base (VCB) calculation method (Los and Duynerke 2000). According to the IPA (Eq. (2)), $A(\tau)$ is equal to the integral of $A(\tau)$ weighted with the normalized $\text{PDF}(\tau)$. $\Delta A$ results from the convexity of $A(\tau)$. Note that if $A(\tau)$ were linear in $\tau$, $\Delta A$ would not occur. The dashed line represents a (fictive) linearized albedo function for $\tau = 18$.

The approximations for the cloud albedo, presented in the next two sections, are applied only to inhomogeneous cloud fields without cloud holes. In subsection 2(b) the albedo function $A$ in the IPA is replaced by a Taylor approximation. In subsection 2(c) a simplified probability distribution function is introduced to permit the analytical integration of the IPA, according to Eq. (2).
Figure 2. Plane-parallel cloud albedo (thick line) and the probability distribution function, PDF, as functions of cloud optical depth, $\tau$, obtained from cloud observations (run R73 of flight RF06). The large dot represents the plane-parallel cloud albedo at the observed mean cloud optical depth, $\bar{\tau} = 18$, and the dashed line gives the respective gradient. The solar zenith angle is 45°.

(b) Taylor approximation of the albedo function

On the basis of the IPA assumption we may rewrite Eq. (2), replacing $A(\tau)$ by the Taylor expansion around the slab-averaged cloud optical depth, $\bar{\tau}$:

$$A(\tau) = A(\bar{\tau}) + \left( \frac{\partial A}{\partial \tau} \right)_{\bar{\tau}} (\tau - \bar{\tau}) + \frac{1}{2} \left( \frac{\partial^2 A}{\partial \tau^2} \right)_{\bar{\tau}} (\tau - \bar{\tau})^2 + \frac{1}{6} \left( \frac{\partial^3 A}{\partial \tau^3} \right)_{\bar{\tau}} (\tau - \bar{\tau})^3 + \cdots .$$

(3)

Replacing $A(\tau)$ in Eq. (2) by the Taylor expansion, (Eq. (3)) yields

$$\overline{A(\tau)} = A(\bar{\tau}) + \frac{1}{2} \left( \frac{\partial^2 A}{\partial \tau^2} \right)_{\bar{\tau}} (\tau^2 - \bar{\tau}^2) + \frac{1}{6} \left( \frac{\partial^3 A}{\partial \tau^3} \right)_{\bar{\tau}} (\bar{\tau}^3 - 3\tau^2\bar{\tau} + 2\bar{\tau}^3) + \cdots .$$

(4)

On the assumption that third and higher-order terms in the Taylor expansion are small (see section 4) we conclude that the reduction in the cloud albedo due to cloud inhomogeneities is determined mainly by the variance of the cloud optical depth, $\overline{\tau^2} = \tau^2 - \bar{\tau}^2$. The approximate IPA in which the albedo is replaced by a truncated Taylor series (referred to as Taylor–IPA) then reads

$$\overline{A_{IPA}(\tau)} = A(\bar{\tau}) + \frac{1}{2} \left( \frac{\partial^2 A}{\partial \tau^2} \right)_{\bar{\tau}} (\tau^2 - \bar{\tau}^2).$$

(5)

The leading term in $\Delta A = \overline{A(\tau)} - A(\bar{\tau})$ is due to the convexity of $A(\tau)$ which means that the second derivative in Eq. (5) is always negative. Because the second moment, i.e. the variance $\overline{\tau^2}$, of the distribution function is strictly positive, the second-order term on the right-hand side of Eq. (5) is negative. Hence, $A(\bar{\tau})$ is reduced by the
value of the second-order term. In section 4 it will be shown that $\Delta A$ of cloud fields with a large cloud optical depth variability is always overestimated due to neglect of third-order and higher-order terms in the Taylor–IPA.

The convex albedo function (see Fig. 2) is always below the tangential line representing a (fictive) linearized albedo function for $\bar{\tau} = 18$ (see Fig. 2). However, $\Delta A$ does not occur for a linear albedo function. Consequently, $\Delta A$ of the inhomogeneous cloud (for which the PDF is given in Fig. 2) is determined by the difference between the linear albedo function and the convex albedo function (Eq. (1)).

The albedo of inhomogeneous cloud fields, determined with the Taylor–IPA (Eq. (5)), is discussed in subsection 3(a) for the bounded cascade cloud field and in subsection 3(b) for a realistic cloud field based on observed microphysical properties (Los and Duynkerke 2000). In section 4 it will be shown that $\Delta A$ of cloud fields with a large cloud optical depth variability is always overestimated due to neglect of third-order and higher-order terms in the Taylor–IPA.

(c) Approximation of the probability distribution function (box function)

According to the IPA (Eq. (2)) $\overline{A(\tau)}$ is obtained by integrating $A(\tau)$ times PDF($\tau$) over $\tau$. The PDF($\tau$) can be obtained from, for instance, cloud models (bounded cascade, see subsection 3(a)) or can be deduced from observations (Los and Duynkerke 2000). However, the PDF($\tau$) of bounded cascade cloud models and the PDF($\tau$) obtained from observations do not agree very well. Cloud optical properties deduced from observed cloud microphysics generate a variety of PDF($\tau$) with no general function, whereas the PDF of the bounded cascade cloud model is typically long tailed towards large $\tau$ and peaks near zero.

The albedo of inhomogeneous cloud fields obtained according to the IPA (Eq. (2)) may be estimated analytically if $A(\tau)$ times PDF is a function which can be integrated over $\tau$. To meet the requirement of integrability the functions $A(\tau)$ and PDF($\tau$) are replaced by the $\delta$-Eddington approximation for conservative scattering, and a box function, respectively.

The normalized variance of the cloud optical depth for a box function of width $2b$ is given by

$$\overline{\tau^2} = \frac{1}{2b} \int_{\bar{\tau} - b}^{\bar{\tau} + b} (\tau - \bar{\tau})^2 \, d\tau = \frac{1}{3} b^2.$$  \hspace{1cm} (6)

For conservative scattering the $\delta$-Eddington approximation of the albedo function is (van Weele and Duynkerke 1993)

$$A_{\delta,\text{Edd}}(\tau) = 1 - \frac{(2 + 3 \mu_0) + (2 - 3 \mu_0) e^{-\tau(1-g^2)/\mu_0}}{4 + 3(1-g)\tau}$$  \hspace{1cm} (7)

where $g$ is the asymmetry factor and $\mu_0$ the cosine of the solar zenith angle. The integration over $\tau$ of $A(\tau)$ weighted with the normalized box function yields

$$\overline{A}_{\mu_0=1}(\bar{\tau}, b) = \frac{1}{2b} \int_{\bar{\tau} - b}^{\bar{\tau} + b} A_{\delta,\text{Edd}}(\tau) \, d\tau = 1 - \frac{1}{6b(1-g)} \times \left\{ (2 + 3 \mu_0) \ln \left( \frac{u}{v} \right) + (2 - 3 \mu_0)(\text{Ei}(u) - \text{Ei}(v)) \exp \left( \frac{4(1-g^2)}{3\mu_0(1-g)} \right) \right\}$$  \hspace{1cm} (8)
where

\[ u = -\frac{[4 + 3(1 - g)(\tau + b)](1 - g^2)}{3\mu_0(1 - g)} \]  

\[ v = -\frac{[4 + 3(1 - g)(\tau - b)](1 - g^2)}{3\mu_0(1 - g)}. \]

The function \( E_i \) represents the exponential integral function (Bronstein and Semendjajew 1987). The consequences of conservative scattering for the calculation of \( \Delta A \) with the box function are discussed in subsection 5(c).

3. MODELLLED AND OBSERVED CLOUD INHOMOGENEITIES

The most realistic cloud optical properties are obtained from \textit{in situ} observations in cloud fields. However, cloud models such as Large Eddy Simulation models, but also the much less computation-intensive bounded cascade cloud models, produce cloud optical properties which are comparable with spectral and statistical properties of observed cloud fields.

For the present investigation the cloud optical properties were obtained with \textit{in situ} observations (Los and Duynkerke 2000) and with a bounded cascade cloud model (Cahalan \textit{et al.} 1994a). In subsection 3(a) the bounded cascade cloud model is introduced and in subsection 3(b) a short description is given of \textit{in situ} cloud observations.

\( \text{(a) Bounded cascade cloud model} \)

The bounded cascade cloud model is presented in several papers, e.g. Cahalan \textit{et al.} (1994a,b). The authors have shown that the bounded cascade cloud model and marine boundary-layer cloud fields have similar spectral properties. Hence, the fractal cloud model is frequently used in studies of cloud radiative properties which involve 2- or 3-dimensional radiative-transfer calculations.

The iterative function of the bounded cascade cloud model reads

\[ q_1^{(n)} = (1 - \rho f c^n) q_1^{(n-1)} \]  

where \( n=0, q_1 = \bar{q}_1 \) is the initial (mean) specific cloud liquid-water content, \( \rho = \pm 1 \), chosen randomly, \( f \) is the fractal parameter (defining the variance of specific liquid-water content, \( q_1 \)), and the parameter \( c = 2^{1/3} \) defines the \(-5/3\) power law of the \( q_1 \) spectrum. The typical shape of PDF(\( \tau \)) is long tailed towards large \( \tau \) and peaks near zero. An approximate expression for the relative standard deviation of the bounded cascade liquid-water content is derived in appendix A. The result reads (Eq. (A.3))

\[ \sigma_\tau^2 = \frac{\langle q^2 \rangle - q_0^2}{q_0^2} \approx f^2 \left( \frac{c^2}{1 - c^2} \right) + f^4 \left( \frac{c^2}{1 - c^2} \right) \left( \frac{c^4}{1 - c^4} \right). \]

Figure 3 shows \( \sigma_\tau \) versus \( f \) as calculated with the approximate expression in Eq. (A.3) and the exact solution of the bounded cascade cloud model. The approximate expression reproduces the modelled variability well for \( f < 0.65 \) which covers the range up to about 100% relative standard deviation in \( q_1 \). The generating process (we use 10 steps, starting with \( n = 1 \)) produces a series of 1024 values for \( q_1 \). The cloud optical
depth, \( \tau \), being the relative parameter in radiative-transfer studies, can be calculated according to the approximation presented by Stephens and Tsay (1990),

\[
\tau = \frac{3}{2} \frac{\text{LWP}}{r_{\text{eff}} \rho_{\text{water}}} \quad \text{with} \quad \text{LWP} = \rho_{\text{air}} \int_{0}^{H} q_{l} \, dz
\]

where LWP is the vertically integrated amount of cloud liquid water, \( r_{\text{eff}} \) is the effective radius of the cloud droplets, and \( H \) is the geometrical cloud thickness.

For typical stratocumulus with \( \bar{\rho}_{l} = 0.25 \, \text{g kg}^{-1} \), \( r_{\text{eff}} = 10 \, \mu\text{m} \), and \( H = 400 \, \text{m} \), we obtain \( \tau \approx 18 \).

(b) Observations in marine boundary-layer cloud fields

In the present investigation we use observations obtained during flights RF06 and RF07 of the ASTEX campaign (FIRE 1992), and during flight A610 of the BICEP campaign. Observed cloud microphysical and optical properties of marine stratocumulus have been investigated by, for instance, Hignett and Taylor (1996) and Los and Duynkerke (2000). In Los and Duynkerke (2000) \( \tau \) was deduced from in-cloud observations of cloud liquid-water content, \( q_{l}(x) \), and cloud liquid-water gradient, \( \Gamma_{1} \). The cloud optical depth \( \tau \) reads

\[
\tau(x) = \frac{3}{2} \kappa \Gamma_{1}^{2/3} H(x)^{5/3}
\]

which is obtained by integrating Eq. (9) in Los and Duynkerke (2000) over \( z \). For the calculation of \( \tau(x) \) only cloud liquid-water probe observations and standard meteorological parameters (temperature and pressure) are required. The cloud geometrical thickness \( H(x) = (h_{1} - h_{b}(x)) \) is determined by correlating the mean vertical gradient of \( \Gamma_{1} \) with the (horizontal) measurements near cloud-top (cloud top \( h_{1} \) is taken constant). Each measurement point (measurement frequency was 1 Hz) represents a 100-metre-wide cloud fragment. The parameter \( \kappa = (9\pi N \rho_{\text{air}}^{2}/(2 \rho_{I}^{2}))^{1/3} \) contains the liquid water.
droplet-number density, $N$, which is assumed constant for the entire cloud field. $\Gamma_1$ is calculated with all in-cloud observations. For flight RF06 the maximum adiabatic value for $\Gamma_1$ is about $2 \text{ g kg}^{-1}\text{km}^{-1}$ which is about twice the mean value.

Hignett and Taylor (1996) and Los and Duynerke (2000) have shown that the variance of cloud optical depth is highly variable. For example, the distribution functions for $\tau$ of two cloud fields, given in Figs. 9(a) and (b) in Los and Duynerke (2000), are highly different. It can be concluded that cloud optical depth in marine stratocumulus is extremely variable and that the distribution functions vary from highly-peaked types to flat bi-modal types. This shows that the variability of cloud optical properties, and hence the PDF($\tau$), in real cloud fields is difficult to predict.

4. RESULTS

The plane-parallel cloud albedo is calculated by means of an accurate doubling-and-adding radiative-transfer model (de Haan et al. 1987). The inhomogeneous cloud albedo is computed with a Monte Carlo simulation model (Los and Duynerke 2000). The Monte Carlo simulations are compared with the results obtained with the IPA, the Taylor–IPA, and the box function. In this section only monochromatic model calculations are shown where absorption is neglected.

The differences between the IPA and the Monte Carlo model are—as expected—well below the statistical uncertainties of the Monte Carlo results. For albedo simulations, the relative standard deviation of the Monte Carlo model obtained with $10^5$ photons is $\approx 1\%$ (Los et al. 1997).

The results shown in this section illustrate the advantages and limits of the Taylor–IPA and the box function for the purpose of the albedo bias parametrization. To show the important role of the higher-order terms in the Taylor expansion and the limits of the Taylor–IPA, $f$ is set to 0.5 and 0.75. Figure 4 gives the albedo for plane-parallel clouds obtained with the doubling-and-adding model and for fractal clouds using the Monte Carlo model and the Taylor–IPA. The solar zenith angle is taken constant ($\theta_0 = 45^\circ$).

The results obtained with the Taylor–IPA are in good agreement with the Monte Carlo simulations for cloud fields with a relative standard deviation, $\sigma_r \approx 0.7$, i.e. for $f = 0.5$, (Fig. 4(a)). However, the accuracy of the Taylor–IPA decreases drastically for $\sigma_r > 0.7$ due to the neglect of higher-order terms in the Taylor expansion. Note that $\sigma_r$ is a function of $f$, as shown in Eq. (4). However, the $\tau^{1/2}$ is the dominant factor in the Taylor–IPA and determines mainly the albedo bias. For very inhomogeneous cloud fields, however, the Taylor–IPA fails to calculate the exact albedo bias $\Delta A$ with the second-moment term $0.5 \times (\partial^2 A/\partial \tau^2)_\tau \times \tau^{1/2}$ only.

Figure 5(a) shows the plane-parallel cloud albedo and the albedo obtained with the fractal cloud field and the box function, calculated for $\sigma_r(\tau) = 0.57$. It shows that the albedo obtained with the box function is smaller than the fractal cloud albedo. Note that the albedo of the box function is persistently smaller than that of the fractal cloud, not only for $\sigma_r(\tau) = 0.57$, shown here, but for all values of $\sigma_r(\tau)$. For $b = \tau$, i.e. the maximum value for $b$, Eq. (6) yields $\sigma_r(\tau) = 1/\sqrt{3}$ which corresponds to the maximum variance of the cloud optical depth obtained with the box function. For $b > \tau$, negative values for $\tau$ would be obtained which is physically unrealistic. Hence, the current box function is only valid for $\sigma_r(\tau) \leq 1/\sqrt{3}$, i.e. $b \leq \tau$.

Figure 5(b) shows the cloud albedo based on run R73, as simulated with the Monte Carlo model, together with the plane-parallel cloud albedo and the albedo obtained with the box function. From cloud observations performed during flight RF06, run R73, $\tau$ is
Figure 4. Albedo (upper panels) of plane-parallel (PP) and fractal clouds (MC, simulated with the Monte Carlo model) as functions of cloud optical depth, $\tau$. The fractal cloud albedo (Taylor–IPA) is represented by stars and the solid line which give the absolute albedo bias. The dashed curve at the bottom of the upper panels gives the plane-parallel albedo bias, $\Delta A$. The solar zenith angle, $\theta_s$, is equal to 45$^\circ$. The results are presented for fractal parameters, $f = 0.5$ and 0.75, in (a) and (b), respectively. The relative difference in albedo, $\delta_{\text{rel}}$, between the Taylor–IPA and the Monte Carlo simulation is shown for the fractal cloud in the lower panel of each figure.

obtained with Eq. (11), i.e. $\tau$ varies in horizontal and vertical directions (the cloud field is determined with the VCB calculation method, see Los and Duynkerke (2000)). The relative standard deviation of $\tau$ is $\sigma_\tau(\tau) = 0.4$. $\Delta A$ obtained with the box function and $\Delta A$ obtained from R73 are shown in Fig. 5(b). The width of the box function was calculated with the observed $\sigma_\tau(\tau)$ according to Eq. (6). Figure 5(b) shows that the cloud albedo based on run R73 is smaller but close to the albedo of the box function.
Figure 5. Albedo as function of cloud optical depth, \( \tau \). (a) The albedo for the plane-parallel cloud (PP), the fractal cloud (MC), and the box function (box). For the fractal cloud and the box function the relative standard deviation, \( \sigma_\tau(\tau) \), is 0.57. (b) The albedo for the plane-parallel cloud (PP), the box function (box), and the Monte Carlo simulation based on observations during run R73 of flight RF96 (R73). According to the observed standard deviation during run R73, which is \( \sigma_\tau(\tau) = 0.4 \), the width of the box function was the same.

However, with the fractal cloud field, \( \Delta A \) would be smaller than \( \Delta A \) of the box function. For large variabilities with \( \sigma_\tau(\tau) \approx 0.5 \) or more, \( \Delta A \) would, therefore, be underestimated appreciably with the fractal cloud model.

From Figs. 5(a) and (b) it can be concluded that the box function performs well and offers an acceptable, yet analytically simple, PDF for estimation of the albedo bias.

5. PARAMETRIZATION OF THE CLOUD OPTICAL DEPTH VARIANCE AND ALBEDO BIAS

The purpose of the cloud albedo parametrization is to include radiative effects of cloud inhomogeneities in climatological and meteorological models, referred to as GCMs. The emphasis is put on marine stratocumulus, without cloud holes, for which the albedo is estimated by reducing the plane-parallel cloud albedo with the albedo bias, \( \Delta A \). So far the parametrization \( \Delta A \) has been given for conservative scattering, i.e. \( \omega_0 = 1 \), only. The implication of conservative scattering for \( \Delta A \) is discussed in subsection 5(c).

It has been shown in the previous subsection that it is the variance of the cloud optical depth, \( \tau^2 \), which mainly determines \( \Delta A \). However, more detailed information about cloud optical properties, such as the PDF(\( \tau \)) (inherently related to \( \tau^2 \)), is not available in numerical models. In order to estimate \( \Delta A \), \( \tau^2 \) must be known. In the next subsection we present a method for parametrizing \( \tau^2 \) as a function of the characteristic length-scale and the mean liquid-water content, \( \bar{q}_l \). The independent variables \( \lambda \) and \( \bar{q}_l \) are calculated in the numerical model for each grid box. The relation between \( \tau^2 \) and \( \bar{q}_l^2 \) is presented in subsection 5(b).

(a) Parametrization of liquid-water variance

Boundary-layer turbulence characteristics are used to account for the variability of specific cloud liquid-water content, \( q_l^2 = (\bar{q}_l^2 - \bar{q}_l^2) \).
In studies of boundary-layer cloud fields the energy spectrum of quasi-conserved quantities, like $q_1$, is used to determine the characteristic length-scales of energy input and energy dissipation. As presented by Davis et al. (1996a), the energy spectrum of $q_1$ fluctuations follows a $-5/3$ power law throughout the inertial subrange. Similar behaviour is noted for the $u$, $v$, and $w$ wind velocity components (east, north, upward-pointing components, respectively). In the inertial subrange the one-dimensional energy spectra of wind velocities $u$, $v$, and $w$, depend on the viscous dissipation rate of turbulence kinetic energy, $\epsilon$, and the wave number, $k = 2\pi/\lambda$,

$$E_u = \alpha \epsilon^{2/3} k^{-5/3}$$
$$E_{v,w} = \frac{4}{3} \alpha \epsilon^{2/3} k^{-5/3}.$$ (12b)

The value of $\alpha \approx 0.55$ is an empirical constant (see, for example, Garrett (1992)).

The energy spectrum of $q_1$ depends on the viscous dissipation rate, $\epsilon$, and the molecular dissipation rate of $q_1$-fluctuations, $\epsilon_{q_1}$. It reads

$$E_{q_1} = \beta \epsilon_{q_1} \epsilon^{-1/3} k^{-5/3}$$ (13)

where $\beta \approx 0.8$ is an empirical constant, taken from Garrett (1992).

Figures 6(a), (b), and (c) show the mean $q_1$-energy spectra for flights RF06, RF07, and the BICEP flight, respectively. Each individual energy spectrum (one per in-cloud run) has been normalized with its variance. The normalized spectra are added to obtain one mean energy spectrum for all in-cloud observations of the respective flights. A correction procedure (Kaimal et al. 1968) has been applied to the spectra to remove the aliasing effect. For flights RF06 and RF07 $q_1$ is derived from the droplet diameters measured with two optical probes (the Forward Scattering Spectrometer Probe (FSSP) and the 260X probe). The FSSP covers the range 4.35–64.25 $\mu$m and the 260X probe covers the range 0–620 $\mu$m. Particles larger than 70 $\mu$m measured with the 260X probe are used only to avoid overlap with FSSP measurements (Los and Duynkerke 2000). During the BICEP flight, $q_1$ is based on the FSSP measurements only. From Fig. 6 one can conclude that the spectra follow approximately the $-5/3$ power law.

The inertial subrange covers the length-scales from several hundreds of metres down to the Kolmogorov scale (about 1 mm). However, there is no clear separation between (3-dimensional) boundary-layer fluctuations and mesoscale fluctuations. Nucciarello and Young (1991) reported that in particular for moisture the mesoscale spectra merge into the inertial subrange. In the convective boundary layer the lack of the spectral gap (Young 1987) is characteristic for the stratocumulus-topped boundary layer (Tjemkes and Visser 1994; Davis et al. 1996a). So far no theoretical explanation has been given for this type of behaviour (Atkinson and Zhang 1996; Jonker et al. 1999). Therefore we have made the assumption that for the determination of $\overline{\rho_{q_1}^2}$ the scaling properties for boundary-layer fluctuations can be extended to mesoscales.

Integrating the energy spectra of an arbitrary scalar, $\phi$, defines the variance $\overline{\rho_{\phi}^2}$ as a function of $k$ which reads

$$\overline{\rho_{\phi}^2}(k) = \int_k^\infty E_{\phi}(k) \, dk.$$ (14)

The characteristic length-scale, $\lambda$, defining the wave number, $k = 2\pi/\lambda$, is chosen to match specified criteria, e.g. if $\lambda$ corresponds to the length of the runs (approximately $60 \, \text{km}$) then $\overline{\rho_{q_1}^2}(k)$ is equal to the variability of the specific liquid-water content of an entire run.
Figure 6. Mean in-cloud energy spectra of specific liquid-water content, $q_l$, as a function of measurement frequency and wavelength. Cloud liquid water was observed with the Forward Scattering Spectrometer Probe (FSSP) and 260X probe during flights (a) RFO6 ($n = 6$) and (b) RFO7 ($n = 6$) and with the FSSP only during (c) the BICEP flight ($n = 4$, where $n$ is the number of spectra used). The solid straight lines indicate the $-5/3$ power law which is in good agreement with the spectra.

In the inertial subrange the energy spectra for $q_l$ (measured at 1 Hz) and $u$, $v$ and $w$ (measured at 20 Hz) yield

$$\overline{q_l^2}(k) = \frac{3}{2} \beta \epsilon q_l \epsilon^{-1/3} k^{-2/3}$$  \hspace{1cm} (15a)

$$\overline{u^2}(k) = \frac{3}{2} \alpha \epsilon^{2/3} k^{-2/3}$$  \hspace{1cm} (15b)

$$\overline{v^2}(k) = \overline{w^2}(k) = 2 \alpha \epsilon^{2/3} k^{-2/3}$$  \hspace{1cm} (15c)

where the spectra of $q_l$ and $u$, $v$ and $w$ are derived as shown by Garratt (1992).

With $\overline{u^2}(k)$, and $\overline{v^2}(k)$, $\overline{w^2}(k)$, the viscous dissipation rate of turbulence kinetic energy, $\epsilon$, is obtained by fitting Eqs. (15b) and (15c) to the observed spectra over
Figure 7. (a) $\epsilon$ and (b) $\epsilon_{q_l}$ (see text) as functions of relative in-cloud height for flights RF06, RF07, and the BICEP flight.

<table>
<thead>
<tr>
<th>Flight</th>
<th>Mean</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(g kg$^{-1}$)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>RF06</td>
<td>0.16</td>
<td>0.09</td>
<td>0.26</td>
</tr>
<tr>
<td>RF07</td>
<td>0.12</td>
<td>0.08</td>
<td>0.17</td>
</tr>
<tr>
<td>BICEP</td>
<td>0.13</td>
<td>0.07</td>
<td>0.17</td>
</tr>
</tbody>
</table>

The inertial subrange with 100 m $< \lambda < 500$ m, where $\lambda = 2\pi/k$. In Fig. 7(a) $\epsilon$ is depicted for flights RF06, RF07 and the BICEP flight. Each symbol represents the value of an in-cloud run. The ensemble average for all runs of all flights yields about $\epsilon = 3 \times 10^{-4}$ m$^2$s$^{-3}$. The molecular dissipation rate, $\epsilon_{q_l}$, is obtained from Eq. (15a) together with $\epsilon$ obtained above. Mean values are depicted in Fig. 7(b). The ensemble average molecular dissipation of $q_l$-fluctuations, $\epsilon_{q_l}$, is about $2 \times 10^{-12}$ s$^{-1}$.

With the assumption mentioned above, the determination of $q_l^{1/2}$ is calculated with $\lambda > 1$ km, while the determination of $\epsilon$ and $\epsilon_{q_l}$ was restricted to $\lambda < 500$ m. For $\lambda = 60$ km the ensemble averaged values ($\epsilon \approx 3 \times 10^{-4}$ m$^2$s$^{-3}$ and $\epsilon_{q_l} \approx 2 \times 10^{-12}$ s$^{-1}$) yield a standard deviation of $\sigma_l = (q_l^{1/2})^{1/2} = 0.12 \times 10^{-3}$ kg kg$^{-1}$ for the specific liquid-water content. The observed standard deviations of $q_l$ are given in Table 1 for flights RF06, RF07 and the BICEP flight. The observed standard deviation corresponds well to the calculated standard deviation of $q_l$.

(b) Relationship between variance in optical depth and liquid-water content

The derivation of $\tau^{1/2}$ as a function of $q_l^{1/2}$ is presented in appendix B. The resulting expression reads (Eq. (B.9))

$$\tau^{1/2} = (2^{2/3} \kappa H q_l^{2/3})^{2} q_l^{1/2}$$

(16)
where $H = (h_1 - h_0)$ is the horizontally averaged geometrical cloud thickness. Table 2 shows the relative standard deviation $\sigma_r = (\tau^2)^{1/2}/\bar{\tau}$ obtained from observations and as calculated with Eqs. (11) and (16). The parameters in Eq. (16) are based on the same observations. Although Eq. (16) is an approximate result for $\tau^{1/2}$ (see appendix B) the agreement is rather good. The agreement between the observed (Eq. (11)) and the calculated $\sigma_r$ (Eqs. (15a) and (16)), which both extend over ranges much larger than the inertial subrange, shows that the spectra of mesoscale fluctuations may be assumed to scale in a similar way as the spectra of boundary-layer fluctuations.

(c) The albedo bias for the solar spectrum

A 4-band radiative-transfer scheme, described by Los and Duynkerke (2000), is used in the doubling-and-adding model and the IPA to calculate the albedo for the solar spectral region. However, for the purpose of the albedo bias parametrization, the restriction $\omega_0 = 1$ (conservative scattering, i.e. no absorption) is required. The consequences of the restriction $\omega_0 = 1$ for the calculation of $\Delta A$ are discussed in this section. Note that the albedo for conservative scattering, i.e. the albedo obtained with the box function, is calculated analytically (Eq. (8)).

Figure 8 (upper panel) shows the albedo calculated with the 4-band scheme as a function of $\tau$ for the plane-parallel cloud and the box function. The inhomogeneous cloud field is calculated for $\sigma_r(\tau) = 0.57$. The albedo functions shown in Fig. 5 are obtained for conservative scattering with the same cloud types. Comparing the conservative albedo (Fig. 5) with the albedo obtained with the 4-band scheme (Fig. 8) reveals that all of the conservative albedo functions are well above the 4-band calculations.

The lower panel of Fig. 8 shows $\Delta A$ calculated with the 4-band scheme for the box function. $\Delta A$ for conservative scattering is shown with a dashed line, labelled ‘$\omega_0 = 1$’. The maximum differences between $\Delta A$ calculated for conservative scattering and $\Delta A$ calculated with the 4-band scheme are about 10%. However, using different probability distribution functions can produce much larger $\Delta A$ differences than the differences between $\Delta A$ obtained for conservative scattering and $\Delta A$ obtained with the 4-band scheme. We conclude that an uncertainty of 10% in the determination of $\Delta A$ introduced by using monochromatic calculations (conservative scattering) instead of spectral calculations is negligible compared with the uncertainties introduced by using the box function.

To give an impression of the strong decrease of $\Delta A$ with increasing $\omega_0$, we depict other curves (dashed lines) showing $\Delta A$ for $\omega_0 = 0.999$, $\omega_0 = 0.99$, and $\omega_0 = 0.9$.

(d) Summary of the parametrization

The parametrization of $\Delta A$ is based on the analytical solution of the cloud albedo, $A_{\omega_0=1}$, presented in section 2, Eq. (8). The independent variables are the mean cloud optical depth, $\bar{\tau}$, the grid size, $\lambda_c$, the mean liquid-water content, $\bar{q}$, and the width of
Figure 8. Upper panel: Albedo calculated with the 4-band scheme as a function of cloud optical thickness, $\tau$, for the plane-parallel cloud (PP-4) and the box function (box-4). For the inhomogeneous cloud field (box-4) the relative standard deviation $\sigma_c(\tau)$ is set to 0.57. Lower panel: albedo bias, $\Delta A$, obtained with the box function for various values of single scattering albedo, $\omega_0$ (dashed lines), and for the 4-band scheme (box-4). The solar zenith angle is set to 45° for all model calculations.

the PDF($\tau$), $b$ (box function). The width $b$ of the box function is obtained by combining Eqs. (15a), (16), and (6). Briefly, $b$ depends on $\lambda_c$ and on $\bar{q}_l$. The grid size $\lambda_c$ replaces $k$ in Eq. (15a) according to $k = 2\pi/\lambda_c$. The resulting expression for $b$ reads

$$b = \sqrt[3]{3^{1/3}}$$

$$= c \left( \frac{\lambda_c}{\bar{q}_l} \right)^{1/3} \text{ with } c = \sqrt{\frac{9}{2}} \beta \epsilon_q \epsilon^{-1/3} \left( \frac{2}{\pi} \right)^{1/3} \kappa H. \quad (17)$$

Using typical values for the parameters $\beta$, $\epsilon_q$, $\epsilon$, $\kappa$, and $H$ as given in Table 3 the constant $c$ is about $4 \times 10^{-2}$ m$^{-1/3}$. The constant $c$ includes the standard deviation of the cloud optical depth, $\sigma_\tau = \sqrt{\tau^2}$. For $\lambda_c = 60$ km, $\bar{q}_l = 0.25$ g kg$^{-1}$, and $c \approx 4 \times 10^{-2}$ m$^{-1/3}$, the standard deviation of cloud optical depth $\sigma_\tau = \sqrt{\tau^2} = (c/\sqrt{3})(\lambda_c/\bar{q}_l)^{1/3}$ is about 14. With $\tau = 18$ (see example in section 3(a)) $\sigma_\tau$ is 80%. It can be concluded that the result of the parametrization is comparable with the observed relative standard deviations given in Table 2.

Finally, $\Delta A$ is obtained according to the definition

$$\Delta A \equiv A_{\omega_0=1}(\tau) - \overline{A}_{\omega_0=1}(\tau, b) \quad (18)$$

using Eqs. (7) and (8).

According to the results presented in section 5(c), $\Delta A$ for conservative scattering is comparable with $\Delta A$ calculated with the 4-band scheme. To account for the plane-parallel albedo bias one has to subtract the parametrized $\Delta A$ obtained with Eq. (18)
from the plane-parallel cloud albedo of the numerical model. Note that the effects of cloud overlap and cloud fraction are not included in the parametrization.

6. SUMMARY AND CONCLUSION

Radiative-transfer calculations overestimate the albedo of plane-parallel clouds compared with the albedo of inhomogeneous clouds, both cloud fields have the same mean cloud optical depth. The difference between the albedo of plane-parallel clouds and inhomogeneous clouds is known as the albedo bias, referred to as $\Delta A$. The purpose of the present investigation is to derive a parametrization for $\Delta A$, based on the variability of the cloud optical depth, which can be employed in climatological and meteorological models (GCMs). As stratocumulus has a large albedo bias and because these cloud fields are frequently observed and last for long periods in subtropical high-pressure regimes, we have based our parametrization on this cloud type only.

For the purpose of the parametrization we have investigated boundary-layer turbulence characteristics and cloud microphysical and macrophysical properties in order to obtain a relationship between the variance of the cloud optical depth and the basic parameters available in GCMs. In section 1 a ‘flow diagram’ is presented which illustrates the sequence of processes used for the parametrization. The variance of the cloud optical depth, $\tau^2$, is obtained from the variance of cloud liquid-water content. The latter is derived from the theory of turbulence, describing spectral properties in a cloud-capped atmospheric boundary layer. With the variance of cloud liquid water we derived the optical properties, i.e. the variance of the optical depth, according to the analysis of cloud macrophysical, microphysical, and optical properties, presented by Los and Duynkerke (2000). Spectral analysis of cloud liquid water, as observed in the stratocumulus-topped boundary layer, shows that cloud liquid-water fluctuations occur on any scales within the inertial subrange. However, there is no clear separation between (3-dimensional) boundary-layer fluctuations and mesoscale fluctuations. To account for mesoscale fluctuations in the parametrization of $\Delta A$, we extended the validity of turbulence properties of marine boundary-layer cloud fields—theoretically defined for the inertial subrange only—to the size of typical GCM grid boxes, which range from 50 to 500 km.

The widely used Independent Pixel Approximation (IPA) is a suitable method for including horizontal cloud inhomogeneities in cloud albedo calculations. The method weights the plane-parallel cloud albedo with the probability density function of the cloud optical depth field. We discuss the estimation of $\Delta A$ for marine boundary-layer cloud fields using two approximative PDFs, first of all with the Taylor–IPA and secondly with a box function replacing the PDF. With the box function we give an analytical solution
of the cloud albedo for conservative scattering. Note that the width of the approximative PDFs is determined with the explicitly calculated variance of the cloud optical depth field.

The albedo bias of horizontally and vertically variable cloud fields is obtained with a Monte Carlo model. The model is initialized with microphysical properties observed in marine stratocumulus and therefore allows us to compare realistic albedo bias simulations with the albedo bias parametrization. We conclude that the results obtained with the box function, for variances of the cloud optical depth obtained from stratocumulus observations, are in acceptable agreement with Monte Carlo simulations, which are based on the same variances.

In general, GCMs calculate only a few cloud parameters, such as the mean cloud liquid-water content, the mean cloud optical depth, and the cloud cover per time step and grid box. With the parametrization proposed in the present paper, the albedo bias is obtained analytically as a function of cloud liquid-water content and grid size, only, which makes the parametrization suitable for use in GCMs.

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APPENDIX A

Bounded cascade cloud model variability

An elaborate description of the higher-order moments of the bounded cascade model is given by Cahalan et al. (1994a). However, as the cascade starts at $n = 1$ (in contrast to Cahalan et al. (1994a) who gave the results for $n = 0$) the second-order term is explicitly given here.

With $q$ representing specific liquid-water content, the standard deviation of the fluctuations is given by $(q^2) = \langle q^2 \rangle - q_0^2$, where $q_0$ denotes the mean liquid-water content.

The bounded cascade cloud model has the iterative function

$$ q_N = \left( \prod_{i=1}^{N} (1 + \rho f e^i) \right) q_0. $$

Note that $q_N = q_N(\rho)$ with $\rho = -1, +1$ where $\rho$ denotes the randomly distributed direction of the cascade steps.

Calculating $\langle q^2 \rangle$ with $q = q_N(\rho)$ where the angle brackets denote the mean value over $\rho$ yields

$$ \langle q_N^2 \rangle = \left\{ \left( \prod_{i=1}^{N} (1 + \rho f e^i)^2 \right) q_0^2 \right\} $$

$$ = q_0^2 \prod_{i=1}^{N} \langle (1 + \rho f e^i)^2 \rangle $$

$$ = q_0^2 \prod_{i=1}^{N} (1 + f^2 e^{2i}). \quad (A.1) $$
Considering the product \( \prod (1 + f^2 c^{2i}) \) one obtains for the first- and second-order terms
\[
\prod_{i=1}^{N} (1 + f^2 c^{2i}) = 1 + \sum_{i=1}^{N} f^2 c^{2i} + \sum_{k=1}^{N-1} \sum_{i=k+1}^{N} f^2 c^{2k} f^2 c^{2i} + \ldots.
\]
In the limit for \( N \to \infty \) the expression for \( \langle q^2_N \rangle \) in Eq. (A.1) becomes
\[
\langle q^2_N \rangle = q_0^2 \prod_{i=1}^{N} (1 + f^2 c^{2i}) = q_0^2 \left( 1 + \sum_{i=1}^{N} f^2 c^{2i} + \sum_{k=1}^{N-1} \sum_{i=k+1}^{N} f^2 c^{2k} f^2 c^{2i} + \ldots \right).
\]
\[
\vdots
\]
\[
= q_0^2 \left[ 1 + f^2 \left( \frac{c^2}{1 - c^2} \right) + f^4 \left( \frac{c^2}{1 - c^2} \right) \left( \frac{c^4}{1 - c^4} \right) + \ldots \right]. \tag{A.2}
\]

The relative standard deviation is defined as \( \sigma_r = \langle q^2 \rangle^{0.5} / q_0 \), and with \( \langle q^2 \rangle = \langle q^2 \rangle - q_0^2 \) one obtains
\[
\sigma_r^2 = \frac{\langle q^2 \rangle - q_0^2}{q_0^2}
\]
\[
= \frac{q_0^2 \left[ 1 + f^2 \{ c^2 / (1 - c^2) \} + f^4 \{ c^2 / (1 - c^2) \} \{ c^4 / (1 - c^4) \} + \ldots \right] - q_0^2}{q_0^2}
\]
\[
\approx f^2 \left( \frac{c^2}{1 - c^2} \right) + f^4 \left( \frac{c^2}{1 - c^2} \right) \left( \frac{c^4}{1 - c^4} \right). \tag{A.3}
\]

\section*{Appendix B}

\textit{Relation between } \( \tau^2 \) and \( q_1^2 \)

Integrating Eq. (9) of Los and Duynkerke (2000) yields \( \tau \) as a function of \( x \),
\[
\tau(x) = \int_{h_b(x)}^{z_h} \beta_e(x, z) \, dz = \int_{h_b(x)}^{z_h} \kappa \Gamma_1^{2/3} (z - h_b(x))^{2/3} \, dz
\]
\[
= \frac{3}{5} \kappa \Gamma_1^{2/3} (h_t - h_b(x))^{5/3}. \tag{B.1}
\]

With the Reynolds decomposition \( \tau(x) = \bar{\tau} + \tau' \) and \( h_t - h_b(x) = h_t - (\bar{h}_b + \bar{h}') \) Eq. (B.1) becomes
\[
\bar{\tau} + \tau' = \frac{3}{5} \kappa \Gamma_1^{2/3} \left( \frac{h_t - (\bar{h}_b + \bar{h}')} \right)^{5/3}. \tag{B.2}
\]

The mean cloud optical depth, \( \bar{\tau} \), is then obtained by averaging Eq. (B.2). The difficulty of averaging the cloud height in Eq. (B.2), labelled cloud height, can be
overcome by expanding the cloud height with a Taylor series

\[
\begin{align*}
\frac{[h_t - (h_b + h_b')]^{5/3}}{[h_t - (h_b + h_b')]^{5/3}} & = (h_t - h_b)^{5/3} \left(1 - \frac{h_b'}{h_t - h_b}\right)^{5/3} \\
& \approx H^{5/3} \left(1 - \frac{5}{3} \frac{h_b'}{h_t - h_b} + \frac{5}{9} \frac{h_b'^2}{(h_t - h_b)^2}\right) \\
& \approx H^{5/3} \left(1 + \frac{5}{9} \frac{h_b'^2}{H^2}\right).
\end{align*}
\] (B.3)

The mean cloud optical depth is then obtained as

\[
\bar{\tau} = \frac{3}{5} \kappa \Gamma_1^{2/3} H^{5/3} \left(1 + \frac{5}{9} \frac{h_b'^2}{H^2}\right).
\] (B.4)

To calculate the variance for \(\tau\) we use the equality \(\bar{\tau}^2 = \bar{\tau}^2 - \tau^2\). The derivation of \(\bar{\tau}^2\) is similar to the previous derivation for \(\bar{\tau}\). The result is

\[
\bar{\tau}^2 = \left(\frac{3}{5} \kappa \Gamma_1^{2/3} H^{5/3}\right)^2 \left(1 + \frac{35}{9} \frac{h_b'^2}{H^2}\right).
\] (B.5)

For \(\tau^2\) we obtained

\[
\tau^2 = \left(\frac{3}{5} \kappa \Gamma_1^{2/3} H^{5/3}\right)^2 \left\{1 + \frac{10}{9} \frac{h_b'^2}{H^2} + \frac{25}{81} \left(\frac{h_b'^2}{H^2}\right)^2\right\}.
\] (B.6)

The difference between Eq. (B.5) and Eq. (B.6) yields

\[
\bar{\tau}^2 = (\kappa \Gamma_1^{2/3} H^{5/3})^2 \frac{\bar{h_b'^2}}{H^2} \left(1 - \frac{1}{9} \frac{\bar{h_b'^2}}{H^2}\right).
\] (B.7)

With the relation \(q_1(x) = 0.5 \Gamma_1(h_t - h_b(x))\) and the Reynolds decomposition \(q_1(x) = q_1 + q_1'\) and \(h_b(x) = \bar{h}_b + h_b'\), the variance \(\bar{h}_b'^2\) is replaced by \(\bar{q}_1'^2\) according to

\[
\overline{q_1'^2} = \left(\frac{1}{2} \Gamma_1\right)^2 \frac{\bar{h}_b'^2}{H^2}
\]

yielding \(\bar{\tau}^2\) as a function of the variance \(\bar{q}_1'^2\)

\[
\bar{\tau}^2 = \left(2^{2/3} \kappa H \bar{q}_1'^2\right)^2 \frac{\bar{q}_1'^2}{\bar{q}_1^2} \left(1 - \frac{1}{9} \frac{\bar{q}_1'^2}{\bar{q}_1^2}\right).
\] (B.8)

With \(\bar{q}_1'^2/\bar{q}_1^2 \ll 1\) (see Table 1) we can approximate Eq. (B.8), yielding

\[
\bar{\tau}^2 = \left(2^{2/3} \kappa H \bar{q}_1'^2\right)^2 \frac{\bar{q}_1'^2}{\bar{q}_1^2}.
\] (B.9)
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